Theory for neoclassical toroidal plasma viscosity in finite aspect ratio tokamaks with broken symmetry*

K. C. Shaing

Institute for Space and Plasma Sciences, National Cheng Kung University, Tainan 70101, Taiwan

I. Introduction

Real tokamaks do not possess toroidal symmetry. When toroidal symmetry is broken, particle, momentum, and heat transport fluxes are all enhanced [1,2]. The toroidal angular momentum is no longer conserved as expected. A comprehensive transport theory has been developed for large aspect ratio tokamaks with broken symmetry. The theory is summarized in [3,4]. The transport fluxes are calculated on the constant pressure surfaces because of the magnetohydrodynamic (MHD) equilibrium. When magnetic surfaces are not broken, i.e., when KAM surfaces are intact, the constant pressure surfaces are the perturbed magnetic surfaces. Thus, on the perturbed magnetic surfaces, parallel electron heat conductivity $\chi_{\|e}$ does not contribute to transport fluxes. The results of the theory are in good agreement with numerical results in the large aspect ratio limit [5-7]. The theory has since been extended to finite aspect ratio tokamaks with broken symmetry [8]. In the superbanana plateau and superbanana regimes, perturbed particle distribution is localized in the phase space. This makes it possible to remove the assumption of the large aspect ratio in these collisionality regimes [8]. Here, we extend the transport fluxes in the collisional boundary layer $\sqrt{\nu}$ regime to include the effects of finite aspect ratio in tokamaks with broken symmetry, where ν is the typical collision frequency. Again, the extension is made possible by utilizing the fact that the boundary layer solution is localized in the phase space.

II. Magnetic Coordinates and $|\boldsymbol{B}|$ Spectrum

We adopt Hamada coordinates [9], where magnetic field $\boldsymbol{B}$ can be expressed as

$$\boldsymbol{B} = \psi' \nabla V \times \nabla\theta - \chi \nabla V \times \nabla\zeta, \tag{1}$$

where V is the volume enclosed inside the magnetic surface divided by $4\pi^2$, θ is the poloidal angle, ζ is the toroidal angle, $\psi' = \boldsymbol{B} \bullet \nabla\zeta$, ψ is the toroidal flux divided by 2π, $\chi = \boldsymbol{B} \bullet \nabla\theta$, χ

is the poloidal flux divided by 2π, and prime denotes d/dV. The $|\boldsymbol{B}|=B$ spectrum on the perturbed magnetic surface can be written as

$$B = B(V,\theta) - B_0 \sum_n \left[A_n(\theta)\cos n\zeta_0 + B_n(\theta)\sin n\zeta_0\right], \quad (2)$$

where $B(V,\theta)$ represents the axisymmetric magnetic field strength on the magnetic surface, ζ_0 = $q\theta$ -ζ is the field line label, $A_n(\theta)$ = $\sum_m \{b_{mnc}\cos[(m-nq)\theta] + b_{mns}\sin[(m-nq)\theta]\}$, $B_n(\theta)$ = $\sum_m \{-b_{mnc}\sin[(m-nq)\theta] + b_{mns}\cos[(m-nq)\theta]\}$, and B_0 is the magnetic field strength on the magnetic axis. The $A_n(\theta)$, and $B_n(\theta)$ terms are the consequences of the broken symmetry in tokamaks. We are interested in the situation that the magnitude of the perturbed magnetic field strength is weak enough so that there are no new classes of trapped particles. Thus, the theory is applicable for rippled tokamaks when ripple trapping is insignificant.

III. Bounce Averaged Drift Kinetic Equation

The transport fluxes in the collisional boundary layer regime are calculated form the solution of the bounce averaged drift kinetic equation. The linear version is [1]

$$\left\langle \mathbf{v}_d \bullet \nabla\zeta_0 \right\rangle_b \frac{\partial f_{01}}{\partial \zeta_0} + \left\langle \mathbf{v}_d \bullet \nabla V \right\rangle_b \frac{\partial f_M}{\partial V} = \left\langle C(f_{01}) \right\rangle_b, \quad (3)$$

where f_{01} is the perturbed distribution function, f_M is the equilibrium Maxwellian distribution, $\mathbf{v}_d$ is the drift velocity, the angular brackets denote the bounce average $\langle \bullet \rangle_b$ = $\oint d\theta(\bullet)/\mathrm{v}_\parallel \boldsymbol{n} \bullet \nabla\theta \Big/ \oint d\theta/\mathrm{v}_\parallel \boldsymbol{n} \bullet \nabla\theta$, $\mathrm{v}_\parallel$ is the particle speed that is parallel to $\boldsymbol{B}$, $\boldsymbol{n}$ is the unit vector in the direction of $\boldsymbol{B}$.

The drift velocity $\mathbf{v}_d$ has the conservative form $\mathbf{v}_d = -\ \mathrm{v}_\parallel \boldsymbol{n} \times \nabla(\mathrm{v}_\parallel/\Omega)$, where Ω is the gyro-frequency. This form for $\mathbf{v}_d$ is valid for low β plasmas. Here β is defined as the ratio of the thermal pressure to the magnetic field pressure. However, it is trivial to show that it is also valid for finite β plasmas in the radial and $\nabla\zeta_0$ directions. It is the $\mathbf{v}_d$ in these two directions that appear in Eq.(3). The expressions for $\left\langle \mathbf{v}_d \bullet \nabla\zeta_0 \right\rangle_b$, and $\left\langle \mathbf{v}_d \bullet \nabla V \right\rangle_b$ can be found in [8].

IV. Solution in the Collisional Boundary Layer Regime and Transport Fluxes

We solve Eq.(3) in the collisional boundary layer regime for tokamaks with broken symmetry. In this regime, the collision frequency is smaller than the toroidal drift frequency. The solution in the region away from the trapped-circulating boundary is

$$f_{01} = \sum_n \frac{McE}{e\chi\langle \mathbf{v}_d \bullet \nabla\zeta_0\rangle_b}\left(\overline{A}_n \cos n\zeta_0 + \overline{B}_n \sin n\zeta_0\right)\frac{\partial f_M}{\partial V}, \quad (4)$$

where $E = \mathrm{v}^2/2$, v is the particle speed,

$$\begin{pmatrix}\overline{A}_n\\ \overline{B}_n\end{pmatrix} = \int_{-\theta_t}^{\theta_t} d\theta \frac{B/B_m}{\left[k^2 - (B/B_m - 1)/(B_M/B_m - 1)\right]^{1/2}}\left[-\lambda\frac{B}{B_m} + 2\left(1 - \lambda\frac{B}{B_m}\right)\right]\times$$

$$\frac{B_0}{B}\begin{bmatrix}A_n(\theta)\\ B_n(\theta)\end{bmatrix}\left\{\int_{-\theta_t}^{\theta_t} d\theta \frac{B/(B_m\varepsilon')}{\left[k^2 - (B/B_m - 1)/(B_M/B_m - 1)\right]^{1/2}}\right\}^{-1}, \quad (5)$$

$k^2 = (1-\lambda)/\left[\lambda(B_M/B_m - 1)\right]$, $\lambda = \mu B_m/E$, and B_M and B_m are respectively the global maximum and minimum of axisymmetric $B(V,\theta)$.

The solution in Eq.(4) does not satisfy the boundary condition and diverges logarithmically at the trapped-circulating boundary where $k^2 = 1$. A boundary layer analysis is needed to make the solution satisfying the boundary condition [10,11]. Because the layer is narrow in the pitch angle space, only pitch angle scattering operator is needed in the boundary layer analysis without the need of the large aspect ratio assumption. The resultant solution that satisfies the boundary condition and matches Eq.(4) when away from the boundary layer is

$$f_{01} = \sum_n \frac{McE}{e\chi}\left(\overline{\alpha}_n \cos n\zeta_0 + \overline{\beta}_n \sin n\zeta_0\right)\frac{\partial f_M}{\partial V}, \quad (6)$$

where

$$\langle \mathbf{v}_d \bullet \nabla\zeta_0\rangle_b^{-1}\begin{pmatrix}\overline{\alpha}_n\\ \overline{\beta}_n\end{pmatrix} = \begin{pmatrix}\overline{A}_n\\ \overline{B}_n\end{pmatrix}\left(1 - e^{-\sqrt{|n|}y}\cos\sqrt{|n|}y\right) + \begin{pmatrix}\overline{B}_n\\ -\overline{A}_n\end{pmatrix}\sigma_w e^{-\sqrt{|n|}y}\sin\sqrt{|n|}y, \quad (7)$$

σ_w denotes the sign of $\langle \mathbf{v}_d \bullet \nabla\zeta_0\rangle_b$, $y = (1-k^2)\left[\Delta(k^2)\right]^{-1}$ is the layer variable, the layer width is

$$\Delta(k^2) = \left\{\frac{4\nu_D}{\lambda^2(1 - B_M/B_m)^2\langle \mathbf{v}_d \bullet \nabla\zeta_0\rangle_b}\frac{\oint d\theta|\mathrm{v}_\parallel/\mathrm{v}|}{\oint d\theta(B/B_M)/(|\mathrm{v}_\parallel/\mathrm{v}|)}\right\}^{1/2}, \quad (8)$$

the pitch angle scattering frequency is $\nu_D = \sum_b \nu_D^{ab}$, $\nu_D^{ab} = \nu_{ab}\left[\Phi(\mathrm{v}/\mathrm{v}_{tb}) - G(\mathrm{v}/\mathrm{v}_{tb})\right]/(\mathrm{v}/\mathrm{v}_{ta})^3$, the the basic relaxation frequency $\nu_{ab} = 4\pi N_b(e_a e_b)^2\ln\Lambda/(\mathrm{v}_{ta}^3 M_a^2)$, N is plasma density, e is the electric charge, M is the mass, v_t is the thermal speed, the subscripts a and b denote plasma species, $\ln\Lambda$ is the Coulomb logarithm, the Chandrasekhar function is $G(x)$ = $\left[\Phi(x)-x\Phi'(x)\right]/\left(2x^2\right)$, and $\Phi(x)$ is the error function.

Using the solution in Eq.(6), the transport fluxes can be evaluated. The Onsager symmetric flux surface averaged radial particle flux Γ and heat flux are

$$\begin{pmatrix} \Gamma \bullet \nabla V \\ \boldsymbol{q} \bullet \nabla V/T \end{pmatrix} = -\frac{1}{4}\left(\frac{Mc}{|e|\chi}\right)^2 \left(\frac{B_M}{B_m}-1\right)^{-1} \int d\mathrm{v}\nu_D \mathrm{v}^6 \frac{\partial f_M}{\partial V} \int_0^1 dk^2 \lambda^{-1} \left(\oint d\theta \left|\frac{\mathrm{v}_{\parallel}}{\mathrm{v}}\right|\right) \times$$

$$\sum_n \left[\left(\frac{\partial \overline{\alpha}_n}{\partial k^2}\right)^2 + \left(\frac{\partial \overline{\beta}_n}{\partial k^2}\right)^2\right]. \qquad (9)$$

From the flux-force relation [12], these fluxes are related to the toroidal plasma viscosity and can be used to model plasma flow in tokamaks with broken symmetry.

V. Conclusions

We have extended theory for neoclassical toroidal plasma viscosity for tokamaks with broken symmetry in the superbanana plateau, superbanana, and collisional boundary layer regimes to include effects of finite aspect ratio and finite β. They can be employed to model plasma flows in finite aspect ratio tokamaks with broken symmetry, invoking flux-force relation.

*This work was supported by Taiwan Ministry of Science and Technology (MOST) under Grant No. 100-2112-M-006-004-MY3.

References

[1] K. C. Shaing, Phys. Plasmas **10**, 1443 (2003).

[2] K. C. Shaing, Phys. Rev. Lett. **87**, 245003 (2001).

[3] K. C. Shaing, M. S. Chu, C.T. Hsu, S. A. Sabbagh, J. Seol, and Y. Sun, Plasma Phys. Control. Fusion **54**, 124033 (2012).

[4] K. C. Shaing, S. A. Sabbagh, and M. S. Chu Nucl. Fusion **50**, 025022 (2010).

[5] Y. Sun, Y. Liang, K. C. Shaing, H. R. Kostowski, C. Wiegmann, and T. Zhang, Phys. Rev. Lett. **105**, 145002 (2010).

[6] Y. Sun, Y. Liang, K. C. Shaing, H. R. Kostowski, C. Wiegmann, and T. Zhang, Nucl. Fusion **51**, 053015 (2011).

[7] S. V. Kasilov, W. Kernbichler, A. F. Martitsch, H. Maassberg, Ma. F. Heyn, Phys. Plasmas **21**, 092506 (2014).

[8] K. C. Shaing, J. Plasma Phys. **81**, 905810203 (2015).

[9] S. Hamada, Nucl. Fusion **2**, 23 (1962).

[10] K. C. Shaing, P. Cahyna, M. Becoulet, J.-K. Park, S. A. Sabbagh, and M. S. Chu, Phys. Plamas **15**, 082506 (2008).

[11] K. C. Shaing, J. Seol, Y. Sun, M. S. Chu, and S. A. Sabbagh, Nucl. Fusion **50**, 125008 (2010).

[12] K. C. Shaing, M. S. Chu, and S. A. Sabbagh, Nucl. Fusion **50**, 125012 (2010).