Perturbation Theory for the Collective Interaction of Relativistic Particles with a Localised EM Wave in a Magnetized Plasma

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1. INTRODUCTION

The interaction of a spatially and temporally localized EM wave with charged particles in a magnetized plasma is investigated for the general case of a Gaussian wave-packet and relativistically moving particles, allowing the consideration of applications related to energetic ions and runaway electrons in fusion as well as in space plasmas. The EM field is introduced as a perturbative term, whereas suitable transformations to Guiding Center (GC) variables are employed to represent the kinetic parameters of the relativistically moving particles. The collective characteristics of the interaction are expressed in terms of ensemble averaged variations of the corresponding canonical momenta.

2. LIE PERTURBATION SCHEME

The unperturbed Hamiltonian of the interaction expressed in GC canonical variables is given from the expression [1], [2] $\mathcal{H}_0 = \sqrt{2mc^2}\sqrt{\Omega P_\phi + P_z^2/2m + mc^2/2}$, where P_ϕ and P_z are the canonical momenta conjugate to the gyro-angle φ and the GC position z along the magnetic field, respectively, with $P_\varphi = (1/2)m\Omega\rho^2$, $\Omega = qB_0/m$ is the cyclotron frequency, ρ is the Larmor radius and m the rest mass of the particle. The first order perturbative term due to the presence of the EM wave is $\mathcal{H}_1 = ec^2(e\mathbf{A}_0 - \mathbf{p})\cdot\mathbf{A}_{em}/E_0$ where $\mathbf{B}_0 = \nabla\times\mathbf{A}_0$ and E_0 is the unperturbed total particle energy.

The EM field is described in terms of the arbitrarily polarized vector potential [3] as $\mathbf{A}_{em} = A_1(\mathbf{r}-\mathbf{V}t,t)[P_{1,\parallel}\hat{\mathbf{z}} + \mathbf{P}_{1,\perp}]e^{i\Psi(\mathbf{r},t)}$ where $\Psi(\mathbf{r},t) = k_\parallel z - \omega t + k_\perp R\sin\theta - k_\perp\rho\sin\varphi = \Psi(\mathbf{R},t) - k_\perp\rho\sin\varphi$ is the phase term of the propagating wave, ω is the central wave frequency, $\mathbf{V}$ is the group velocity, $\mathbf{R}$ the GC position vector and $k_\parallel$, $k_\perp$ are the wave-vector components with respect to $\mathbf{B}_0$. Applying a Fourier expansion, the first order perturbation term of the Hamiltonian will now become

$$\mathcal{H}_1 = \frac{ec^2}{E_0}e^{i\Psi(\mathbf{R},t)}\sum_n e^{-in\varphi}A_1(\mathbf{r}-\mathbf{V}t,t)\left[u_{0\parallel}P_\parallel J_n(k_\perp\rho) + u_{0\perp}\left(P_+J_{n+1}(k_\perp\rho) + P_-J_{n-1}(k_\perp\rho)\right)\right] \quad (1)$$

where $P_{||}$, P_+, P_- are the corresponding parallel, right-hand and left-hand circular polarizations of the vector potential and J_n are Bessel functions. For the vector-potential envelope carrying the spatiotemporal modulation of the wave-packet, a Gaussian-shaped configuration is adopted as being suitable for analytical calculations, that is $A_1(\mathbf{r}-\mathbf{V}t,t)=A_1\,\mathrm{e}^{-\left[\frac{(x-V_xt)^2+(y-V_yt)^2}{a_\perp^2}+\frac{(z-V_zt)^2}{a_{||}^2}\right]-\frac{t^2}{a_\tau^2}}$, where $a_\perp, a_{||}$, and a_τ represent the spatial and the temporal waist of the Gaussian. Applying now a standard Lie-transform canonical perturbation method [4], the first order Lie generating function is

$$w_1=-\int_{t_0}^{t}\mathcal{H}_1(\boldsymbol{\varphi}+\boldsymbol{\omega}_{\boldsymbol{\varphi}}(\tau-t),\mathbf{J},\tau)d\tau \tag{2}$$

where $\mathbf{J}=(P_z, P_\varphi, P_\theta)$, $\boldsymbol{\varphi}=\boldsymbol{\omega}_{\boldsymbol{\varphi}}t+\boldsymbol{\varphi}_0$, and $\boldsymbol{\omega}_{\boldsymbol{\varphi}}=\partial\mathcal{H}_0/\partial\mathbf{J}$ are the corresponding actions, angles and frequencies for the unperturbed system $\mathcal{H}_0$, whereas $P_\theta=(1/2)m\Omega R_\perp^2$ and $\boldsymbol{\varphi}_0$ denotes the initial gyrophase. Moreover, taking the Taylor expansion of the Gaussian and keeping terms up to first order with respect to $\rho/a_\perp$, results in the expression

$$\begin{aligned} w_1=\frac{-ec^2\sqrt{\pi}}{E_0}\tau\sum_n F_n\, e^{i\mathbf{R}\cdot(\bar{\mathbf{k}}+\tau^2\Delta_n\mathbf{V})}\, e^{-in\varphi}\Big[G_n-\frac{\rho}{a_\perp}\left(G_{n+1}e^{i\theta}+G_{n-1}e^{-i\theta}\right)\frac{R}{a_\perp}+\\ +\frac{\rho}{a_\perp}\tau^2\left(\mathbf{R}\cdot\mathbf{V}+i\frac{\Delta_n}{2}\right)\frac{V_\perp}{a_\perp}\left(G_{n+1}e^{i\xi}+G_{n-1}e^{-i\xi}\right)\Big]+c.c \end{aligned} \tag{3}$$

where $G_n=u_{0||}P_{||}J_n(k_\perp\rho)+u_{0\perp}P_+J_{n+1}(k_\perp\rho)+u_{0\perp}P_-J_{n-1}(k_\perp\rho)$, $u_{0||}, u_{0\perp}$ are the components of the mechanical momentum, $\xi=\tan^{-1}(V_y/V_x)$. The function F_n is defined as

$$F_n=e^{-\tau^2|\mathbf{R}\times\mathbf{V}|^2}\, e^{-|\mathbf{R}|^2\frac{\tau^2}{a_\tau^2}}\, e^{-\frac{\tau^2}{4}\Delta_n^2} \tag{4}$$

where $\mathbf{R}=\left[\frac{X}{a_\perp},\frac{Y}{a_\perp},\frac{z}{a_{||}}\right]$, $\mathbf{V}=\left[\frac{V_x}{a_\perp},\frac{V_y}{a_\perp},\frac{V_z-P_z/\gamma m}{a_{||}}\right]$, $\bar{\mathbf{k}}=\left[0,k_\perp a_\perp,k_{||}a_{||}\right]$ and $\tau=\frac{1}{\sqrt{\mathbf{V}^2+1/a_\tau^2}}$ is the autocorrelation time of the wave-packet as seen by the particles. Moreover,

$$\Delta_n=k_{||}P_z/(\gamma m)-\omega-n\Omega/\gamma \tag{5}$$

with $\gamma=E_0/mc^2$ whereas $\Delta_n=0$ provides the resonance condition [2], [3]. The function F_n describes the most important features of the particle interaction with the localized wave-packet. The first term of (4) describes the dependence of the interaction on the relation between the wave-packet group velocity and the particle GC position as in a scattering-like process, whereas the second term describes the time duration of the interaction. The third term describes the resonance mechanism of the interaction and reveals its localized effect on the action-space of the GC motion, that is centered around $\Delta_n = 0$ and has a characteristic width inversely proportional to τ.

3. AVERAGE VARIATIONS OF ACTIONS

The average variation of the actions for an ensemble of particles with uniformly distributed canonical angles is given from [2], [4]

$$\langle \delta P_\ell \rangle = \frac{1}{2}\frac{\partial}{\partial P_\ell}\left\langle \left(\frac{\partial w_1}{\partial \ell}\right)^2 \right\rangle_{z,\varphi,\theta} \tag{6}$$

where ℓ=(z, φ, θ,) and provides the average variation of parallel momentum (P_z), Larmor radius ($\rho = (2P_\varphi / m\Omega)^{1/2}$), and transverse GC position ($R_\perp = (2P_\theta / m\Omega)^{1/2}$). In this short paper, we present results only for parallel propagation of the EM wave ($\mathbf{k}_\perp = 0$) for both parallel and circular polarization of the EM wave. In the following figures, we depict the normalized averaged variations of each action $\Delta P_\ell = \langle \delta P_\ell \rangle / (A_1 \tau)^2$ as functions of the parallel momentum P_z for wave-packets with different characteristics, whereas the resonance condition $\Delta_n = 0$, is written in the form

$$P_\varphi = [P_z^2(k_\parallel^2 - \omega^2) - \omega^2 + n^2 - 2nk_\parallel P_z]/2\omega^2 \tag{7}$$

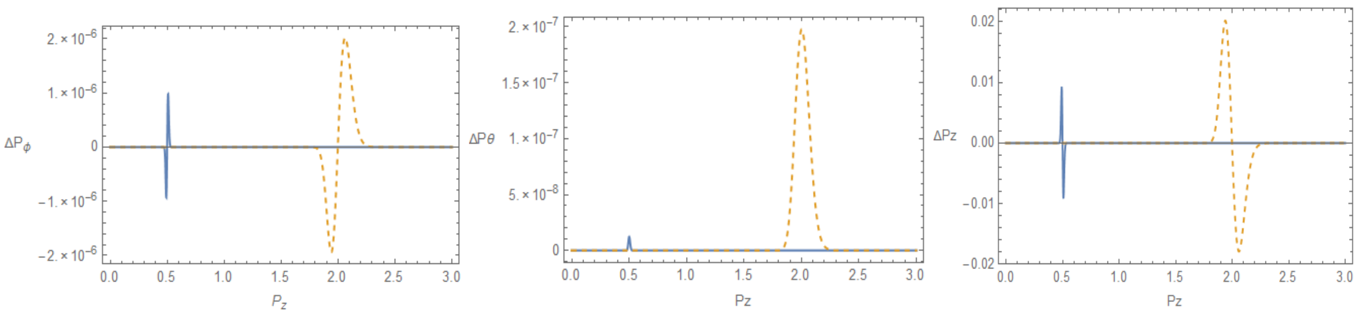

Figure 1:Variations of the averaged actions for $P_\parallel = 1, P_\theta = 1, k_\parallel = 1, \omega = 1, \mathbf{V} = 0, a_\parallel = a_\perp = 10^2, a_\tau = 10^3$, $P_\varphi = 0.5$ (solid line) and $P_\varphi = 2$ (dashed line). In this special case with $k_\parallel = \omega$, the coefficient of the quadratic term in (7) is eliminated, and the resonance condition Δ_n=0 is fulfilled only for $n \neq 0$. Expression (3) suggests that only the terms with $n = \pm 1$ ($n = -1$ for $P_z > 0$) are nonzero, a restriction that is imposed by Bessel-functions' properties. This is a physical feature that is due to the finite Larmor-radius effect $\sim \rho / a_\perp$ introduced in our analysis. Moreover, by reducing the transverse spatial extent of the wave packet $a_\perp$,we have seen that both ΔP_φ and ΔP_θ are rather enhanced, (while the width of the resonances is preserved) whereas ΔP_z remained unchanged.

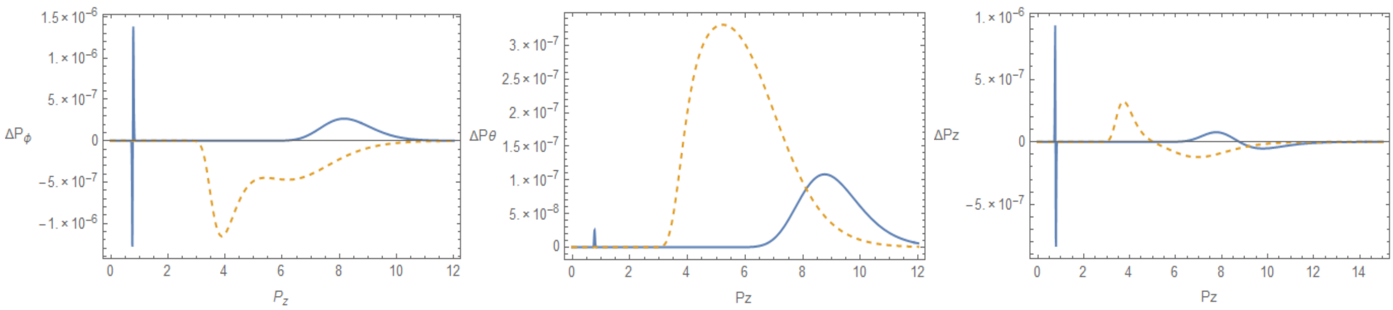

Figure 2: Variations of the averaged actions for $P_\parallel = 1, P_\theta = 1, k_\parallel = 1, \omega = 1.1, \mathbf{V} = 0, a_\parallel = a_\perp = 10^2, a_\tau = 10^3$, $P_\varphi = 0.5$ (solid line) and $P_\varphi = 2$ (dashed line). In this non-degenerate case with $\omega \neq k_\parallel$, the resonance condition between P_z and P_φ has a parabolic form, which results in two resonant curves, each with different width. Since $k_\parallel < \omega$, it is also easy to check that both resonant curves correspond to $n = -1$. As P_φ is approaching the peak of the parabola, the resonant curves tend to overlap; clearly, beyond that point, the resonance will be infeasible.

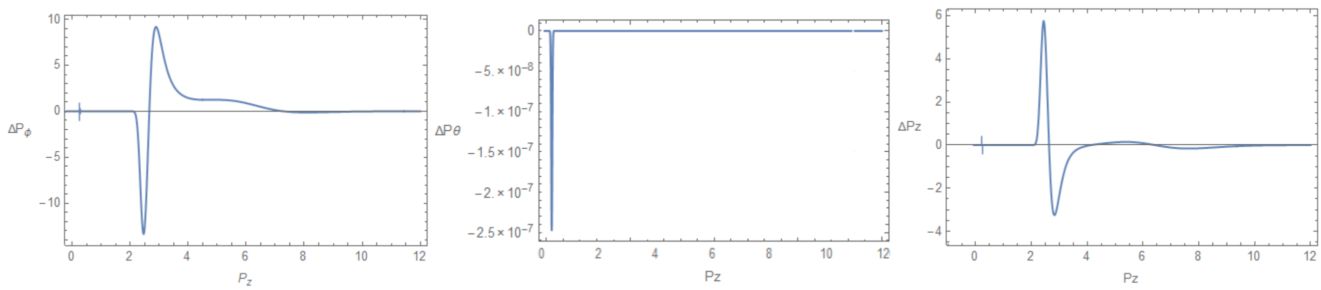

Figure 3:Variations of the averaged actions for $P_\parallel = 0, P_+ = 1, P_\theta = 1, k_\parallel = 1, \omega = 1.1, \mathbf{V} = 0, a_\parallel = a_\perp = 10^2, a_\tau = 10^3$ $P_\varphi = 1.5$. Introducing a circularly (e.g right-hand) polarized EM wave ($P_\parallel = P_- = 0$) we observe that the existence of higher-order Bessel terms may enrich the spectral content of the interaction. Once again, $n \neq 0$ for $k_\parallel < \omega$, and from Bessel function properties the resonances correspond to $n = \{-2, -1\}$. For both ΔP_φ and ΔP_z an overlap between the resonant curves is present, whereas at the same time the amplitude of ΔP_θ is negligible and only a single resonance is observable, suggesting that the interaction with a circularly polarized wave does not affect considerably the position of the GC.

Work supported in part by the Hellenic National Programme on Controlled Thermonuclear Fusion associated with the EUROfusion Consortium

4. REFERENCES

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