The Spectral Web:
A new theory of the stability of stationary plasma flows

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Abstract

The development of MHD stability theory of laboratory and astrophysical plasmas with sizeable background flows, initiated long ago, has severely suffered from the wide spread misunderstanding that this theory necessarily involves non-self-adjoint operators. The new theory of *the Spectral Web*, on the contrary, is entirely constructed on the basis of the two quadratic forms of the potential energy and the averaged Doppler–Coriolis shift, both involving a self-adjoint operator. This approach finally provides order in the bewildering variety of complex eigenvalues that are obtained from high resolution spectral codes. This order obtains from a monotonicity property of the eigenvalues in the complex plane along the two sets of curves that constitute the spectral web. Such a property was long thought to be restricted to the real eigenfunctions of static equilibria, but it has now been generalized for the complex eigenfunctions of stationary equilibria. The monotonicity can not be based on node counting of the eigenfunctions, but it involves a quantity called *the complementary energy* which represents the energy needed to sustain the Doppler–Coriolis shifted oscillations of the instabilities. Thus, the full complex spectrum of stationary plasmas is obtained together with a connecting structure. This permits to consider the enormous diversity of MHD instabilities of laboratory and astrophysical plasmas with arbitrary flow and rotation profiles from a single unifying viewpoint. The method is illustrated with old and new instabilities of a force free field equilibrium subjected to shear flow.

1. The Spectral Web method

Spectral theory of magnetohydrodynamic (MHD) waves and instabilities of magnetized plasmas has mainly been developed for static equilibria. For example, most "intuition" on tokamak stability derives from the energy principle for static plasmas, which has been the standard stability paradigm for over half a century [1]. However, most plasmas in magnetic fusion devices have substantial flows, and in astrophysics the paradigm simply makes no sense because there are no static plasmas in the Universe. The required modification for stationary equilibria has been known since the appearance of the seminal paper by Frieman and Rotenberg [2] in 1960, but, unfortunately, further development of spectral theory along this line has been hampered by the general misunderstanding that this theory necessarily involves non-self-adjoint operators. The new approach, exploiting what will be called the *Spectral Web*, is based on the opposite observation, viz. that the Frieman–Rotenberg spectral equation,

$$\mathbf{G}(\xi) - 2\rho\omega U\xi + \rho\omega^2\xi = 0\,, \tag{1}$$

is a non-linear eigenvalue problem involving *two self-adjoint operators*, viz. the generalized force operator $\mathbf{G}$ and the Doppler–Coriolis operator $U \equiv -\mathrm{i}\mathbf{v}_0 \cdot \nabla$. With these operators two real quadratic forms may be associated, viz. the solution averages of the potential energy W and of the Doppler–Coriolis shift V of the perturbations. For eigenvalues, $\sigma \equiv \mathrm{Re}(\omega) = \overline{V}$.

The usual proof of self-adjointness of the force operator is modified by exploiting, for arbitrary values of ω, the left solution ξ^{ℓ} of Eq. (1) that satisfies the BCs on axis and the right solution ξ^{r} that satisfies the BCs at the wall. The two solutions are joined at some surface S inside the plasma. At that surface, the surface energy $W_{\rm com}$ is constructed,

$$W_{\rm com} = -\tfrac{1}{2}\int \xi_n^* [\![\Pi(\xi)]\!]\, dS\,, \qquad (2)$$

where the normal component ξ_n is made continuous and the jump $[\![\Pi(\xi)]\!]$ of the total pressure perturbation does not vanish, in general. This *complementary energy* represents the amount of energy to be injected or extracted at S to obtain exponential time behavior $\exp(-\mathrm{i}\omega t)$. Self-adjointness implies that $W_{\rm com}$ vanishes, which is the case for eigenvalues. Instead, in the Spectral Web method, the real and imaginary parts of $W_{\rm com}$ are contour plotted to obtain two sets of curves in the complex ω-plane on which one of the two vanishes:

$$\mathrm{Im}[W_{\rm com}(\omega)] = 0 \;\Rightarrow\; \textit{solution path}\,, \qquad \mathrm{Re}[W_{\rm com}(\omega)] = 0 \;\Rightarrow\; \textit{conjugate path}\,. \qquad (3)$$

The eigenvalues are located at the intersections of these two curves. Thus, for the first time, the general eigenvalue problem of stationary equilibria is solved by an intuitive method that not only provides the complex eigenvalues of the instabilities, but also connects them with physically meaningful curves. The method is applied to the instabilities of a straight cylinder, for which the reduction of the Frieman–Rotenberg equation (1) is well documented [3, 4]. The Spectral Web method is not restricted to cylindrical problems though since reduction to ordinary differential equations in the normal direction for ξ_n and Π also obtains for toroidal problems where the tangential dependences are taken care of by a separate reduction. The present form of the method is a complete revision of earlier papers on the subject [5].

2. An example: Instabilities of a force-free magnetic field

As an example illustrating the method of the Spectral Web, we choose the classical problem of the stability of a force-free field (FFF) in cylindrical geometry. The equilibrium is given by the well-known Bessel function model with constant ratio α between the current density and the magnetic field. This problem first arose in astrophysical context [6]. The stability analysis of this configuration requires careful study of the rational surface singularities according to Newcomb's theorems [7]. That analysis was carried out by Voslamber and Callebaut [8], whereas the corresponding eigenfunctions and growth rates were calculated by Goedbloed and Hagebeuk [9]. The problem of the stability of force-free magnetic fields of constant α is important for the mechanism of relaxation and magnetic reconnection [10], as applied to reversed field pinches [11], spheromaks [12], but also magnetic structures in the solar corona [13].

Instability with respect to ideal internal kink modes obtains for sufficient current density, $\alpha a \geq 3.175$. The fastest growing eigenmodes then exhibit one lobe ($n = 1$) in the radial coordinate. Increasing the value of α, an increasing number of eigenmodes (with an increasing number n of lobes) would become unstable, if it were not for Newcomb's fundamental stability theorem which dictates that the stability of a plasma with rational surfaces should be investigated as if it consisted of independent nested sub-domains bounded by the rational surfaces. This corresponds to increased stability, due to the stabilizing influence of surface currents flowing at the rational surfaces. This stabilization mechanism disappears for resistive plasmas, so that the higher n modes are resistively unstable. For example, for $\alpha a \geq 6.843$, the $n = 1$ mode is an unstable ideal internal kink mode (with approximately constant amplitude in each independent sub-interval in between the rational surfaces, but rapidly varying across, see Fig. 5 of Ref. [9]), whereas the $n = 2$ mode is ideally stable, but unstable with respect to tearing.

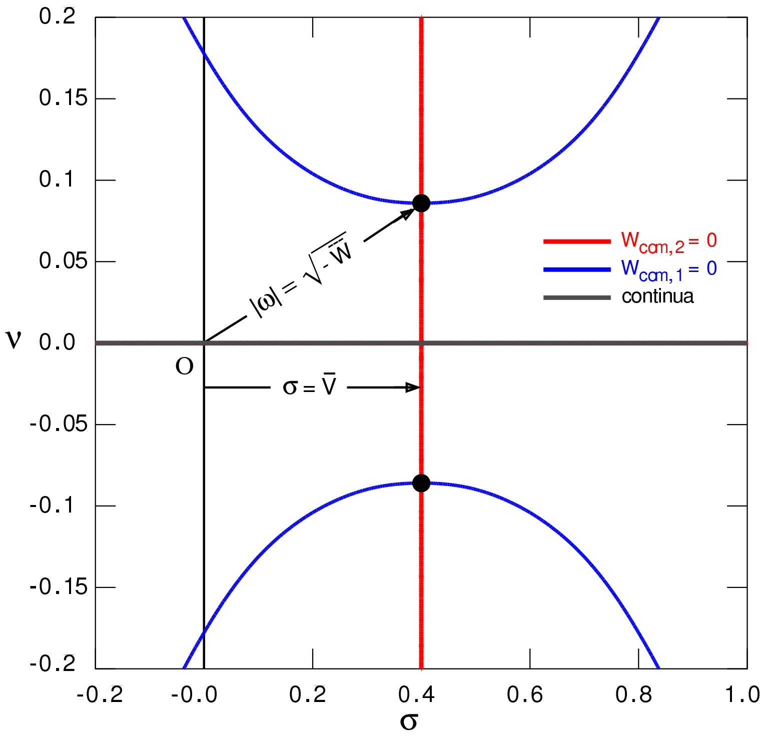

Figure 1: *Spectral Web for the* $m=1$, $k=1.28$ *internal kink mode of a FFF of constant* $\alpha = 8.0$, *subjected to a constant Doppler shift,* $kv_z = 0.4$. *Eigenvalues (black dots) are located at the intersections of the solution path (red) and the conjugate path (blue).*

In Fig. 1 the spectral web is shown for the internal kink mode of a FFF equilibrium subjected to a constant flow in the longitudinal direction, a trivial change with respect to the static equilibrium. The solution path is just a straight line parallel to the imaginary axis, and the mode has the same growth rate as in the static case, but it obtains a finite real oscillation frequency (the Doppler shift). The value of α is chosen such that there are two rational surfaces in the plasma. The effect of *shear flow* onto the spectral web is shown in Fig. 2: A second branch of the solution path appears with a faster growing mode on it (labeled I_2): the $n=2$ tearing mode has become ideally unstable! Moreover, a tiny structure of intersecting solution path and conjugate path curves appears with modes labeled G_1, $G_2, \ldots$ on it (further illustrated in the inset). These modes result from the extrema of the continuous spectra (plotted versus $x \equiv r/a$ in the upper part of the figure). Those extrema would give rise to stable Global Alfvén Eigenmodes (GAEs) on a small sub-interval about the extrema (really local modes then), but on the full interval they become unstable with a global eigenfunction. We will call them Flow-driven Global Alfvén Eigenmode (FGAE) instabilities. There is an infinity of them.

In summary: the Spectral Web method has revealed, in a very direct and intuitive way, the existence of new modes: a flow-driven $n=2$ internal kink mode, with localization *outside* the first singularity, and a class of infinitely many FGAEs with localization across the whole plasma but with rapid oscillations at the extrema of the continua.

3. Conclusions

- *The Spectral Web is a powerful new tool producing the complete complex spectrum of instabilities of plasma equilibria with flow.*
- *The break-up in inner/outer plasma regions is optimal for the incorporation of external modes, fast particle effects, control of instabilities by RF excitation, etc.*
- *The example of a FFF illustrates how shear flow turns tearing modes into ideal internal kink modes, and induces Flow-driven Global Alfvén Eigenmode (FGAE) instabilities.*
- *Many more examples have been computed (interchanges, resistive wall modes, Rayleigh–Taylor, magneto-rotational instabilities, etc.): to be reported elsewhere.*

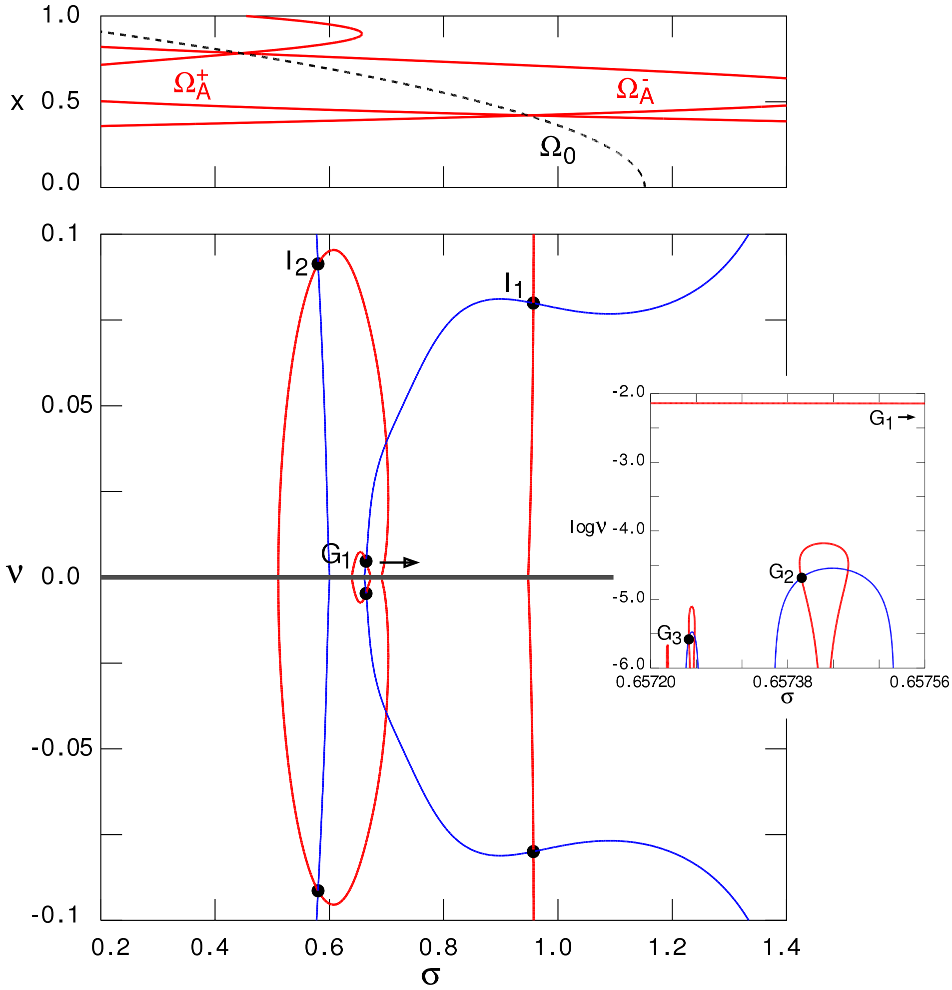

Figure 2: *Spectral Web for the different modes of the same force-free magnetic field configuration as in Fig.1, but subjected to shear flow,* $v_z = 0.9(1-r^2)$*. Two internal kink modes (*I_1 *and* I_2*) and an infinity of FGAEs (*G_1*,* G_2*, ...) are indicated in the inset. The upper part of the figure indicates the radial profiles of the Doppler shift* Ω_0 *and of the Doppler shifted Alfvén and slow continua* $\Omega_A^{\pm}$ *and* $\Omega_S^{\pm}$*.*

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