Transport theory with effects of finite banana width in tokamaks with broken symmetry*

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Real tokamaks are not toroidally symmetric. A transport theory has been developed for large aspect ratio tokamaks with broken symmetry [1]. The theoretical predictions are in agreement with numerical results in the $\varepsilon < 1$ limit, with ε, the inverse aspect ratio [2.3]. The theory hasbeen extended to include effects of finite ε, and finite β [4,5], with β, the ratio of the plasma pressure to magnetic field pressure. However, there are cases where the radial wavelength of the perturbed magnetic field can be comparable to $\varepsilon^{1/2}\rho_{pi}$, the width of the bananas [6], where ρ_{p_i} is the ion poloidal gyro-radius, the theory is further extended here to accommodate those short wavelength modes. The relevant equation is the orbit averaged drift kinetic equation to account for the physics of finite $\varepsilon^{1/2}\rho_{pi}$ in the low collisionality regimes. The key assumptions are $\rho_i/L < \rho_{p_i}/L < 1$, but $k_\chi|\nabla\chi|\sqrt{\varepsilon}\rho_{p_i} \sim 1$, with ρ_i, the ion gyro-radius; L, the radial equilibrium scale length; k_χ, the radial wave vector of perturbed modes; and χ, the poloidal flux. The transport fluxes in th superbanana plateau regime are calculated by solving the orbit averaged drift kinetic equation.

In Hamada coordinates, the magnetic field $\boldsymbol{B} = \psi' \nabla V \times \nabla\theta - \chi \nabla V \times \nabla\zeta$, where V is the volume inside the magnetic surface, θ is the poloidal angle, ζ is the toroidal angle, $\psi' = \boldsymbol{B} \bullet \nabla\zeta$, ψ is the toroidal flux, $\chi = \boldsymbol{B} \bullet \nabla\theta$, and prime denotes d/dV. Also $\boldsymbol{B} = G\nabla\theta + F\nabla\zeta + \nabla\varphi$, where F is the poloidal current outside a magnetic surface, G is the toroidal current inside a magnetic surface, φ satisfies $\boldsymbol{B} \bullet \nabla\varphi = B^2 - \langle B^2 \rangle$, and $\langle \bullet \rangle$ denotes flux surface average. The $|\boldsymbol{B}| = B$ spectrum on the perturbed magnetic surface can be written as $B = B_s(V,\theta) - B_0 \sum_n \left[A_n(V,\theta)\cos n\zeta_0 + B_n(V,\theta)\sin n\zeta_0\right]$, where B_s is the axisymmetric magnetic field strength, B_0 is B at the axis, $\zeta_0 = q\theta - \zeta$, q is the safety factor, and n, the toroidal mode number. We assume that the magnitudes of A_n, and B_n are too weak to trap particles, however, allow the radial wavelength of A_n, and B_n comparable to $\varepsilon^{1/2}\rho_{pi}$.

When $\nu_* = \nu Rq/(\mathrm{v}_t\varepsilon^{3/2}) < 1$, transport processes are dominated by the drift dynamics of bananas wobbling off the magnetic surface. Here, R is the major radius, $\mathrm{v}_t = \sqrt{2T/M}$, T is the temperature, and M is the mass. In $(p_\xi, \zeta_0, \theta, E, \mu)$ coordinates, the drift kinetic equation is

$$\dot{\theta}\,\partial f/\partial\theta + \dot{\zeta}_0\,\partial f/\partial\zeta_0 + \dot{p}_\xi\,\partial f/\partial p_\xi = C(f), \qquad (1)$$

where f is the particle distribution, $p_\xi = \chi - (F - \partial\varphi/\partial\zeta_0)\mathrm{v}_\parallel/\Omega$, $\mathrm{v}_\parallel$ is the particle speed parallel to $\boldsymbol{B}$, Ω is the gyro-frequency, $C(f)$ is the Coulomb collision operator, $E = \mathrm{v}^2/2 + e\Phi/M$, $\mu = \mathrm{v}_\perp^2/(2B)$, v is the particle speed, Φ is the electrostatic potential, e is the electric charge, $\dot{\theta} = (\mathrm{v}_\parallel\boldsymbol{n} + \mathbf{v}_d)\bullet\nabla\theta$, $\boldsymbol{n} = \boldsymbol{B}/B$, $\mathbf{v}_d = -\mathrm{v}_\parallel\boldsymbol{n}\times\nabla(\mathrm{v}_\parallel/\Omega)$ is the drift velocity, $\dot{\zeta}_0 = \mathbf{v}_d\bullet\nabla\zeta_0 = (\mathrm{v}_\parallel/B)\,\partial(\mathrm{v}_\parallel\boldsymbol{B}\times\nabla\zeta_0\bullet\nabla\theta/\Omega)/\partial\theta + [\mathrm{v}_\parallel/(\Omega\chi)]\,\partial(\mathrm{v}_\parallel B)/\partial V$, $\dot{p}_\xi = \chi\,\mathbf{v}_{d1}\bullet\nabla V$, and $\mathbf{v}_{d1}\bullet\nabla V = [\mathrm{v}_\parallel/(\Omega\chi)]\,\partial(\mathrm{v}_\parallel B)/\partial\zeta_0$. Only the first order terms in ρ_{p_i}/L ordering are kept in $\dot{p}_\xi$. Note that $\mathbf{v}_d$ in the ∇V, and $\nabla\zeta_0$ directions are also valid for finite β plasmas [5,7].

We adopt the ordering where the bounce frequency ω_b is much larger than the drift frequency ω_d, and collision frequency ν. The leading order equation is $\dot{\theta}\,\partial f_0/\partial\theta = 0$, where $f_0 = f_0(p_\xi, \zeta_0, E, \mu)$ is the zeroth order distribution. The next order equation is

$$(\mathrm{v}_\parallel\boldsymbol{n} + \mathbf{v}_d)\bullet\nabla\theta\,\partial f_1/\partial\theta + \mathbf{v}_d\bullet\nabla\zeta_0\,\partial f_0/\partial\zeta_0 + \chi\,\mathbf{v}_{d1}\bullet\nabla V\,\partial f_0/\partial p_\xi = C(f_0). \qquad (2)$$

Because f_1 is periodic, the following orbit averaged drift kinetic equation must be satisfied

$$\langle\mathbf{v}_d\bullet\nabla\zeta_0\rangle_{ob}\,\partial f_0/\partial\zeta_0 + \chi\,\langle\mathbf{v}_{d1}\bullet\nabla V\rangle_{ob}\,\partial f_0/\partial p_\xi = \langle C(f_0)\rangle_{ob}, \qquad (3)$$

where $\langle\bullet\rangle_{ob} = \oint d\theta(\bullet)B/\omega\chi' / \oint d\theta\,B/\omega\chi'$ denotes the orbit average, and $\omega = (\mathrm{v}_\parallel\boldsymbol{n} + \mathbf{v}_d)\bullet\nabla\theta/\boldsymbol{n}\bullet\nabla\theta$. Because $\rho_{p_i}/L < 1$, $\omega \approx \mathrm{v}_\parallel$ in $\langle\bullet\rangle_{ob}$, and the difference between p_ξ and V on the equilibrium quantities is neglected.

We decompose A_n and B_n as $(A_n \quad B_n) = (A_{0n} \quad B_{0n}) + (A_{1n} \quad B_{1n})$, where A_{0n} and B_{0n} have radial variations of the order of L, and A_{1n} and B_{1n} have much shorter variations than L. We express $(A_{1n} \quad B_{1n}) = \sum_l (A_{1nl} \quad B_{1nl})e^{ilk_\chi\chi}$, where l is a nonzero integer, and A_{1nl} and B_{1nl} are Fourier coefficients. The contributions of A_{0n} and B_{0n} to fluxes can be found in [1]. Here, we present fluxes caused by A_{1n} and B_{1n}. In $(p_\xi, \theta, \zeta_0, E, \mu)$, $\chi = p_\xi + \sigma x\rho_{b\chi}\sqrt{\lambda}\,K(\theta)$ by neglecting $\partial\varphi/\partial\zeta_0$, where $\sigma = \mathrm{sgn}(\mathrm{v}_\parallel)$, $x = (\mathrm{v}/\mathrm{v}_t)$, $\lambda = \mu B_m/E$, $\rho_{b\chi} = (F\mathrm{v}_t/\Omega_m)\sqrt{(B_M/B_m - 1)}$, $\Omega_m = \Omega(B_m)$, $K(\theta) = (B_m/B)\left[k^2 - (B/B_m - 1)/(B_M/B_m - 1)\right]^{1/2}$, k^2

$= (1-\lambda)\Big/\Big[\lambda\big(B_M/B_m-1\big)\Big]$, $k^2>1$ for circulating particles, $k^2<1$ for trapped particles, and B_M and B_m are respectively the maximum and minimum values of $B_s(V,\theta)$.

The toroidal drift frequency is $\langle \mathbf{v}_d \bullet \nabla\zeta_0\rangle_{ob} = c\Phi'/\chi + (McE/e\chi)\,\varepsilon'\,G(k)$, where $\varepsilon' = d\varepsilon/dV$ is a normalization factor. The dimensionless resonance function $G(k)$ is

$$G(k)=\frac{\lambda\omega_{nb}}{\varepsilon'}\int_{\theta_{t1}}^{\theta_{t2}} d\theta K^{-1}(\theta)\left[2\left(\frac{B_M}{B_m}-1\right)\left(k^2-\frac{B_s/B_m-1}{B_M/B_m-1}\right)\left(\frac{\chi'}{\chi}-\left(B_s^2\right)'\Big/\left(2B_s^2\right)\right)+\frac{B_s}{B_m}\left(B_s^2\right)'\Big/\left(2B_s^2\right)\right],$$

where $\omega_{nb} = 1\Big/\int_{\theta_{t1}}^{\theta_{t2}} d\theta K^{-1}(\theta)$, the integration limits are turning points at which $v_{\parallel}(\theta_j) = 0$ for $j = 1$ and 2, and $\chi'/\chi = \langle B^2\rangle'\big/\langle B^2\rangle + 4\pi P'\big/\langle B^2\rangle - Fq'\chi'\big/\langle B^2\rangle$, and P is total pressure. The $\langle \mathbf{v}_{d1}\bullet\nabla V\rangle_b = (McE/e\chi)\sum_{nl}\left[\overline{A}_{1nl}(-n\sin n\zeta_0)+\overline{B}_{1nl}(n\cos nn\zeta_0)\right]e^{ilk_\chi p_\xi}$, where $\omega_{nb}^{-1}\left(\overline{A}_{1nl}\quad \overline{B}_{1nl}\right)=$ $\left\{\int_{\theta_{t1}}^{\theta_{t2}} d\theta K^{-1}(\theta)\left[-\lambda B_s/B_m+2(1-\lambda B_s/B_m)\right](B_0/B_s)\left[A_{1nl}(\theta)\quad B_{1nl}(\theta)\right]\cos\left[lk_\chi\rho_{b\chi}\sqrt{\lambda}xK(\theta)\right]\right\}$.

Effects of finite $\varepsilon^{1/2}\rho_{pi}$ appear in the argument of the cosine function.

We expand Eq.(3) using local transport ordering, i.e., $\nu \sim\langle \mathbf{v}_d\bullet\nabla\zeta_0\rangle_{ob} >$ $\mathbf{v}_{d1}\bullet\nabla V\big/(|\nabla V|L)$. The leading order equation is $\langle \mathbf{v}_d\bullet\nabla\zeta_0\rangle_{ob}\,\partial f_{00}/\partial\zeta_0 = \langle C(f_{00})\rangle_{ob}$, where the second subscript denotes the ordering. The approximate solution is $f_{00} = f_M(V)$, a Maxwellian distribution. The next order equation for f_{01}, the correction to f_{00}, is

$$\langle \mathbf{v}_d\bullet\nabla\zeta_0\rangle_{ob}\,\partial f_{01}/\partial\zeta_0+\langle \mathbf{v}_{d1}\bullet\nabla V\rangle_{ob}\,\partial f_M/\partial V = \langle C(f_{01})\rangle_{ob}, \qquad (4)$$

Because the width of orbits is finite, the definitions for transport fluxes become

$$\begin{pmatrix}\langle\Gamma\bullet\nabla V\rangle_\chi\\ \langle \boldsymbol{q}\bullet\nabla V/T\rangle_\chi\end{pmatrix}=\int\frac{d\chi}{\Delta\chi}\int\frac{d\theta}{2\pi}\int\frac{d\zeta}{2\pi}\int d\mathbf{v}\mathbf{v}_d\cdot\nabla V\begin{pmatrix}1\\ x^2-5/2\end{pmatrix}f, \qquad (5)$$

where Γ is the particle flux, $\boldsymbol{q}$ is the heat flux, and $\langle\bullet\rangle_\chi$ denotes the flux surface and radial averages. The radial average is performed over a region that is wider than $\varepsilon^{1/2}\rho_{pi}$ but narrower than L. Because we order $\sqrt{\varepsilon}\rho_{p_i}\,|\nabla\chi|\sim\lambda_{k_\chi} = 2\pi/k_\chi < |\nabla\chi|L$, averaging over a region wider than $\varepsilon^{1/2}\rho_{pi}$ is the same as averaging over a wavelength. Thus, we choose $\Delta\chi = \lambda_{k_\chi}$.

In the superbanana plateau regime, fluxes are dominated by the resonant particles that have $\langle \mathbf{v}_d\bullet\nabla\zeta_0\rangle_{ob} = 0$. We expand $f_{01} = \sum_{nl}(a_{nl}\cos n\zeta_0 + b_{nl}\sin n\zeta_0)e^{ilk_\chi p_\xi}$, and approximate $\langle C(f_{01})\rangle_{ob} = -\,\nu f_{01}$, where a_{nl} and b_{nl} are Fourier coefficients. Substituting f_{01} into Eq.(4), and taking $\left|v e\chi'/\left(McE\varepsilon'\right)\right| \to 0$ limit yield the resonant part of the Fourier coefficients: $\left(a_{nl}\right)_r$ = - $R_n\left(McE/e\chi\right) n\,\overline{B}_{1nl}\,\partial f_M/\partial V$, and $\left(b_{nl}\right)_r$ = $R_n\left(McE/e\chi\right) n\,\overline{A}_{1nl}\,\partial f_M/\partial V$, where R_n = $\left(\pi/|n|\right)\left[\delta\left(k^2-k_r^2\right)/\left|\partial G/\partial k_r^2\right|\left|McE\varepsilon'/\left(e\chi'\right)\right|\right]$, δ is the delta function, and k_r^2 is the resonant pitch angle parameter at which $\left\langle \mathbf{v}_d \bullet \nabla\zeta_0\right\rangle_{ob} = 0$. Using resonant part of f_{01} in Eq.(5) yields

$$\begin{pmatrix}\left\langle \Gamma \bullet \nabla V\right\rangle_\chi \\ \left\langle \boldsymbol{q} \bullet \nabla V/T\right\rangle_\chi\end{pmatrix} = - \frac{N\mathrm{v}_t^2}{4\sqrt{\pi}\varepsilon'}\frac{Mc}{|e|\chi}\left(B_M/B_m-1\right)^{1/2}\left[\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\left(\frac{p'}{p}+\frac{e\Phi'}{T}\right)+\begin{pmatrix}\eta_2\\ \eta_3\end{pmatrix}\frac{T'}{T}\right], \qquad (6)$$

where N is density, and p is plasma pressure. The coefficients η_j for j = 1 – 3 in Eq.(6) are

$$\eta_j = \int_{x_m}^{\infty} dx x^4\left(x^2-5/2\right)^{j-1} e^{-x^2}\left[1+k_r^2\left(B_M/B_m-1\right)\right]^{-2}\int_{\theta_{t1}}^{\theta_{t2}} d\theta \frac{\lambda_{k_r^2}^{-1/2} B/B_m}{\left[k_r^2-\left(B/B_m-1\right)/\left(B_M/B_m-1\right)\right]^{1/2}} \times$$

$$|n|\left(\left|\overline{A}_{1nl}\right|^2+\left|\overline{B}_{1nl}\right|^2\right)_{k_r^2}\Big/\left|\partial G/\partial k_r^2\right|, \qquad (7)$$

where quantities with the subscript k_r^2 are evaluated at resonant k_r^2. Superbanana plateau resonance can occur only for particles that have normalized speed greater than $x_{\min}$ [1]. The lower limit $x_{\min}$ in Eq.(7) depends on sign of the parameter $\sigma_{\Phi' e}$; $\sigma_{\Phi' e}$ = +1 if Φ' and e have the same signs, otherwise $\sigma_{\Phi' e}$ = -1. If $\sigma_{\Phi' e}$ = 1, $G(k)$ must be negative to have the resonance, and $x_{\min}^2 = 2\left(c|\Phi'|/\chi\right)\left(|e|\chi/Mc\right)\left(\mathrm{v}_t^2\varepsilon'\right)^{-1}\left|G_m(k)\right|^{-1}$, with $G_m(k)$, the global minimum of $G(k)$. If $\sigma_{\Phi' e}$ = -1, $G(k)$ must be positive to resonate, and $x_{\min}^2 = 2\left(c|\Phi'|/\chi\right)\left(|e|\chi/Mc\right)\left(\mathrm{v}_t^2\varepsilon'\right)^{-1}\left|G_M(k)\right|^{-1}$, with $G_M(k)$, the global maximum of $G(k)$.

In conclusion, short wavelength variations of the magnetic perturbations are smoothed out by finite values of $\varepsilon^{1/2}\rho_{pi}$. The radial profile for fluxes varies on the equilibrium scale, consistent with results in [6]. The theory can be extended for energetic alpha particles.

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