Properties of the MHD force operator in the presence of a resistive wall

A.A. Galyuzov and V.D. Pustovitov

Moscow Institute of Physics and Technology, Dolgoprudny, Moscow Region, Russia

National Research Center "Kurchatov Institute", Moscow, Russia

1. Introduction. It is known [1-4] that in the ideal magnetohydrodynamics (MHD) the normal modes are either purely oscillating or purely growing/damped due to the self-adjointness of the ideal MHD force operator. This fact was proved in [1-4] for the plasma surrounded by an ideally conducting wall. Our aim is to investigate the properties of this operator in the presence of a resistive wall, so that the boundary conditions will be the main factor affecting the result. We do it by following the method described in [4], but now with the energy dissipation in the resistive wall which is the main difference of our work from [4].

2. Derivations. A *static* (with mass velocity $\mathbf{V}_0 = \mathbf{0}$) *toroidal magnetically confined ideally conducting* plasma (its volume is denoted as "*pl*") surrounded at some distance with a resistive wall (with its inner surface "*wall-*") is considered. There is a vacuum gap ("*gap*") between the plasma and the wall.

We start from the equation of energy transfer in ideal (no energy dissipation) MHD (for example, equation (5.39) in [3])

$$\frac{\partial}{\partial t}\left(\frac{\rho \mathbf{V}^2}{2}+\frac{p}{\Gamma-1}+\frac{\mathbf{B}^2}{2}\right)+\nabla\cdot\left(\frac{\rho \mathbf{V}^2\mathbf{V}}{2}+\frac{\Gamma}{\Gamma-1}p\mathbf{V}+\mathbf{E}\times\mathbf{B}\right)=0, \tag{1}$$

where ρ is the plasma density, $\mathbf{V}$ is its mass velocity, $\mathbf{B}(\mathbf{E})$ is the magnetic (electric) field, p is the plasma pressure, $\mathbf{j}=\nabla\times\mathbf{B}$ is the current density, Γ is the ratio of specific heats.

Integration of (1) over the volume enclosed by the toroidal wall yields

$$\frac{\partial E}{\partial t} = -\oint\limits_{wall-} \mathbf{E}\times\mathbf{B}\cdot d\mathbf{S}_w^-, \text{ where} \tag{2}$$

$$E = \int\limits_{pl+gap} u dV \text{ with } u = \frac{\rho \mathbf{V}^2}{2}+\frac{p}{\Gamma-1}+\frac{\mathbf{B}^2}{2}. \tag{3}$$

Consider small displacements $\boldsymbol{\xi}(\mathbf{r_0},t)=\mathbf{r}-\mathbf{r_0}$ of the plasma from its equilibrium position $\mathbf{r_0}$. Then the perturbation of the full energy $\delta E = E(\mathbf{r}_0+\boldsymbol{\xi})-E(\mathbf{r}_0)$ may be expanded to the second order in $\boldsymbol{\xi}$ and $\dot{\boldsymbol{\xi}}$ (hereinafter $\dot{f}\equiv\partial f/\partial t$) and be presented as

$$\delta E(\boldsymbol{\xi},\dot{\boldsymbol{\xi}}) = K(\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}})+M(\boldsymbol{\xi},\dot{\boldsymbol{\xi}})+\delta W(\boldsymbol{\xi},\boldsymbol{\xi}), \tag{4}$$

where
$$K(\dot{\boldsymbol{\xi}},\dot{\boldsymbol{\xi}}) = \frac{1}{2}\int\limits_{pl}\rho_0\dot{\boldsymbol{\xi}}^2 dV, \tag{5}$$

$\delta W(\xi,\xi)$ is a functional quadratic in ξ, $M(\xi,\dot{\xi})$ is a functional bilinear in $\xi,\dot{\xi}$, and $\mathbf{b}$ is the perturbation of the magnetic field (the magnetic energy $0.5\int_{gap}\mathbf{b}^2dV$ is included in $\delta W(\xi,\xi)$).

The terms linear in ξ do not appear in (4) because $\nabla E(\mathbf{r}_0)=\mathbf{0}$ in equilibrium.

According to (2) a relation is valid

$$\frac{\partial}{\partial t}\delta E(\xi,\dot{\xi}) = -F_w^-, \text{ where } F_w^- = \int_{wall-}(\mathbf{E}_1\times\mathbf{b})\cdot d\mathbf{S}_w^- \tag{6}$$

and $\mathbf{E}_1$ is the electric field perturbation. In the standard stability theory with an ideal wall, $\mathbf{n}_w\times\mathbf{E}_1=0$ and $F_w^-=0$ ($\mathbf{n}_w$ is the unit normal to the inner surface of the wall). We assume the wall resistive. Then $F_w^-\neq 0$, depending on $\mathbf{E}_1$ and $\mathbf{b}$ at the wall. These quantities are related to $\mathbf{E}_1=-\dot{\xi}\times\mathbf{B}_0$ and $\mathbf{b}=\nabla\times[\xi\times\mathbf{B}_0]$ in the plasma through the boundary conditions at the plasma surface: $\mathbf{n}_{pl}\times\mathbf{E}_1=-(\mathbf{n}_{pl}\cdot\dot{\xi})\mathbf{B}_0$ and $\mathbf{n}_{pl}\cdot(\mathbf{b}_{gap}-\mathbf{b}_{pl})=0$. This coupling has a consequence that $\mathbf{b}=\mathbf{0}$ everywhere at $\xi=\mathbf{0}$, but, maybe, $\dot{\mathbf{b}}\neq\mathbf{0}$, if $\dot{\xi}\neq\mathbf{0}$ at this moment. When $\dot{\xi}=\mathbf{0}$, we have $\mathbf{E}_1=\mathbf{0}$, though, maybe, $\mathbf{b}\neq\mathbf{0}$. This, in particular, means that $F_w^-=0$ at either $\xi=\mathbf{0}$ or $\dot{\xi}=\mathbf{0}$. We use this property below.

With the usage of (5), the left-hand side of the energy balance (6) can be written in the extended form as

$$2K(\dot{\xi},\ddot{\xi})+M(\dot{\xi},\dot{\xi})+M(\xi,\ddot{\xi})+\delta W(\xi,\dot{\xi})+\delta W(\dot{\xi},\xi) \tag{7}$$

As explained above, it must be zero at either $\xi=\mathbf{0}$ or $\dot{\xi}=\mathbf{0}$. Therefore, $M(\dot{\xi},\dot{\xi})=M(\xi,\ddot{\xi})=0$. The logic of the proof is the same as in [4], and the presence of F_w^- in (6) does not spoil it. Let us add that $\ddot{\xi}$ and ξ are related by

$$\rho_0\ddot{\xi}=\mathbf{F}_S(\xi), \tag{8}$$

the standard equation of small oscillations that nullifies the first-order variation of (1).

Then relation (6) reduces to

$$2K(\dot{\xi},\mathbf{F}_S(\xi)/\rho_0)+\delta W(\xi,\dot{\xi})+\delta W(\dot{\xi},\xi)=\int_{wall-}(\dot{\mathbf{A}}\times\nabla\times\mathbf{A})\cdot d\mathbf{S}_w^-, \tag{9}$$

where we introduced the vector-potential by $\mathbf{E}_1=-\dot{\mathbf{A}}$ so that $\mathbf{b}=\nabla\times\mathbf{A}$.

The combination $\delta W(\xi,\dot{\xi})+\delta W(\dot{\xi},\xi)$ in (9) is invariant with respect to the replacement of the arguments ξ and $\dot{\xi}$, and, accordingly, $\mathbf{A}$ and $\dot{\mathbf{A}}$. Therefore,

$$2K(\dot{\xi},\mathbf{F}_S(\xi)/\rho_0)-2K(\xi,\mathbf{F}_S(\dot{\xi})/\rho_0)=\int_{wall-}\left[\dot{\mathbf{A}}\times\nabla\times\mathbf{A}-\mathbf{A}\times\nabla\times\dot{\mathbf{A}}\right]\cdot d\mathbf{S}_w^- \quad \text{or}$$

$$\int_{pl}\dot{\xi}\cdot\mathbf{F}_S(\xi)dV-\int_{pl}\xi\cdot\mathbf{F}_S(\dot{\xi})dV=\int_{wall-}(\dot{\mathbf{A}}\times\nabla\times\mathbf{A}-\mathbf{A}\times\nabla\times\dot{\mathbf{A}})\cdot d\mathbf{S}_w^-, \tag{10}$$

where we have used the consequence of (5) and (8): $2K(\boldsymbol{\eta},\mathbf{F}_S(\boldsymbol{\xi})/\rho_0)=\int_{pl}\boldsymbol{\eta}\cdot\mathbf{F}_S(\boldsymbol{\xi})dV.$

The equality (10) consists of 2 functionals bilinear in $\boldsymbol{\xi}$, $\dot{\boldsymbol{\xi}}$ and $\mathbf{A}$, $\dot{\mathbf{A}}$. Thus we may substitute the pair $\dot{\boldsymbol{\xi}}$, $\dot{\mathbf{A}}$ in (10) for any arbitrary vector fields $\boldsymbol{\eta}(\mathbf{r},t)$ and $\mathbf{Q}(\mathbf{r},t)$, belonging to the same vector space as $\boldsymbol{\xi}(\mathbf{r},t)$ with $\mathbf{A}(\mathbf{r},t)$. As a result, we will have

$$\int_{pl}\boldsymbol{\eta}\cdot\mathbf{F}_S(\boldsymbol{\xi})dV-\int_{pl}\boldsymbol{\xi}\cdot\mathbf{F}_S(\boldsymbol{\eta})dV=\int_{wall-}(\mathbf{Q}\times\nabla\times\mathbf{A}-\mathbf{A}\times\nabla\times\mathbf{Q})\cdot d\mathbf{S}_w^-. \tag{11}$$

It can be derived directly from relations (8.43)-(8.44) of [3] setting there $\mathbf{n}_w\times\mathbf{A}\neq\mathbf{0},\mathbf{n}_w\times\mathbf{Q}\neq\mathbf{0}$. If the wall is ideally conducting, i. e. $\mathbf{n}_w\times\mathbf{A}=\mathbf{0},\mathbf{n}_w\times\mathbf{Q}=\mathbf{0}$, the right-hand side of (11) is zero. Then (11) gives a conventional result [1-4] – self-adjointness of $\mathbf{F}_S$.

For a resistive wall with $\mathbf{j}_1=\sigma\mathbf{E}_1$, equation (11) can be transformed in (σ is the conductivity of the wall)

$$\int_{pl}\boldsymbol{\eta}\cdot\mathbf{F}_S(\boldsymbol{\xi})dV-\int_{pl}\boldsymbol{\xi}\cdot\mathbf{F}_S(\boldsymbol{\eta})dV=\int_{wall}\sigma\left[(\dot{\mathbf{Q}}\cdot\mathbf{A})-(\dot{\mathbf{A}}\cdot\mathbf{Q})\right]dV=0. \tag{12}$$

We can introduce a complex displacement and vector-potential by $\boldsymbol{\xi}=\boldsymbol{\xi}_R+i\boldsymbol{\xi}_I$, $\mathbf{A}=\mathbf{A}_R+i\mathbf{A}_I$, and the same for $\boldsymbol{\eta}$ and $\mathbf{Q}$. We demand that the real and imaginary parts of these complex functions belong to the same vector space as $\boldsymbol{\xi}$ and $\mathbf{A}$. Substitution of such complex functions in (12) does not violate this equality. This means that (11) and, consequently, (12) are valid for complex vector-functions.

Now, substituting in (12) complex vectors $\boldsymbol{\eta}=\boldsymbol{\xi}^*$, $\mathbf{Q}=\mathbf{A}^*$ with * denoting complex conjugation and assuming the time dependence $\propto\exp(\gamma t)$ with $\gamma=\gamma_R+i\gamma_I$, we obtain

$$\int_{pl}\boldsymbol{\xi}\cdot\mathbf{F}_S(\boldsymbol{\xi})dV-\int_{pl}\boldsymbol{\xi}\cdot\mathbf{F}_S(\boldsymbol{\xi})dV=-2i\gamma_I\int_{wall}\sigma|\mathbf{A}|^2dV. \tag{13}$$

Now let us assume that $\gamma_I\neq0$. (14)

Then it follows for $\boldsymbol{\xi}\propto\exp(\gamma t)$ from (8) that

$$\int_{pl}\boldsymbol{\xi}\cdot\mathbf{F}_S(\boldsymbol{\xi})dV-\int_{pl}\boldsymbol{\xi}\cdot\mathbf{F}_S(\boldsymbol{\xi})dV=4i\gamma_R\gamma_I\int_{pl}\rho_0|\boldsymbol{\xi}|^2dV.$$

This relation along with (13) gives $\gamma_R=-\frac{1}{2}\int_{wall}\sigma|\mathbf{A}|^2dV\Big/\int_{pl}\rho_0|\boldsymbol{\xi}|^2dV<0.$ (15)

This result ("a static ideal toroidal magnetically confined plasma surrounded by a resistive wall is **always** stabilized") is absolutely unrealistic. Moreover, for an ideally conducting wall the right hand side of (15) gives a wrong result $\gamma_R=0$ instead of $\gamma_R\gamma_I=0$ [1-3]. It proves that our assumption (14) was wrong and $\gamma_I=0$ for an ideal toroidal plasma surrounded by a resistive wall.

Now it is clear that

$$\int_{pl} \boldsymbol{\eta}\cdot\mathbf{F}_S(\boldsymbol{\xi})dV - \int_{pl} \boldsymbol{\xi}\cdot\mathbf{F}_S(\boldsymbol{\eta})dV = 0, \qquad (16)$$

$$\text{when } \mathbf{n}_w \times \mathbf{E}_1 \neq \mathbf{0} \text{ and } \boldsymbol{\xi}, \boldsymbol{\eta} \propto \exp(\gamma t),\ \cos((\mathbf{k}\cdot\mathbf{r})-\omega t),\ \sin((\mathbf{k}\cdot\mathbf{r})-\omega t) \qquad (17)$$

It can be also obtained that the growth rate must satisfy the quadratic equation [5]

$$\gamma_R^2 \int_{pl} \rho_0 |\boldsymbol{\xi}|^2 dV + \gamma_R \int_{wall} \sigma |\mathbf{A}|^2 dV + C = 0, \text{ where } C \text{ is real.}$$

A small displacement of the perturbed plasma from its equilibrium trajectory in the presence of its equilibrium rotation with a mass velocity $\mathbf{V}_0$ is described by the Frieman-Rotenberg equation [6, 7]

$$\rho_0 \frac{d^2\boldsymbol{\xi}}{dt^2} = \mathbf{F}_S(\boldsymbol{\xi}) + \nabla\cdot(\boldsymbol{\xi}\rho_0(\mathbf{V}_0\cdot\nabla)\mathbf{V}_0) \equiv \mathbf{F}(\boldsymbol{\xi}),$$

where $\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{V}_0\cdot\nabla)$, $\mathbf{F}_S(\boldsymbol{\xi})$ is a volume force density without and $\mathbf{F}(\boldsymbol{\xi})$ - with the plasma equilibrium rotation. It is easy to check that an equation

$$\int_{V_{pl}} \boldsymbol{\eta}\cdot\mathbf{F}(\boldsymbol{\xi})dV - \int_{V_{pl}} \boldsymbol{\xi}\cdot\mathbf{F}(\boldsymbol{\eta})dV = \int_{V_{pl}} [\boldsymbol{\eta}\cdot\nabla\cdot(\boldsymbol{\xi}\rho_0(\mathbf{V}_0\cdot\nabla)\mathbf{V}_0) - \boldsymbol{\xi}\cdot\nabla\cdot(\boldsymbol{\eta}\rho_0(\mathbf{V}_0\cdot\nabla)\mathbf{V}_0)]dV \neq 0$$

is valid ($\boldsymbol{\eta}\neq\boldsymbol{\xi}$) for the perturbation time dependencies listed in (17). This means that equilibrium plasma rotation brings non-self-adjointness into the force operator.

3. Conclusion. It has been proved that the force operator of an ideal plasma surrounded by a resistive wall and displaced from its position of static equilibrium is self-adjoint for most commonly used perturbation time dependencies (17). In a general case, this property of the force operator of a static plasma is determined only by how the perturbation varies in time. The plasma equilibrium rotation that makes the force operator of an ideal plasma explicitly non-self-adjoint is needed to make γ (real for a plasma with $\mathbf{V}_0 = \mathbf{0}$) complex.

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