Normal and reversed magnetic shear Tokamak equilibria with sheared flows of arbitrary direction

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Introduction. In this study we employ a linear model of Grad-Shafranov equilibria with incompressible flows of arbitrary direction, in order to examine the influence of the current column displacements on the equilibrium and stability properties of Tokamak configurations with normal and reversed magnetic shear. An interesting observation is that of magnetic phase transitions resulting in states with coexistent diamagnetic and paramagnetic regions, in both normal and reversed magnetic shear configurations, when the current column is shifted appropriately. These transitions are due to the variation of the fraction of the force-free current density with respect to the diamagnetic one, which is related to the helicity injection inside the torus. We observed also that small displacements of the current column may result in equilibria possessing regions with both normal and reversed magnetic shear. The shear within these regions can be enhanced or reduced by adjusting the triangularity of the configuration.

In terms of the stability properties, the normal and reversed magnetic shear equilibria differ upon the localization and the extent of the stable regions, stable in the sense that a sufficient stability condition is satisfied therein. The results indicate that the reversed magnetic shear can play a stabilizing role.

Equilibrium solutions. The computed equilibria are derived by analytical integration of the generalized Grad-Shafranov (GGS) [1],[2] equation in cylindrical coordinates (r,ϕ,z) with z the axis of symmetry,

$$\partial_{rr}u - r^{-1}\partial_r u + \partial_{zz}u + \frac{1}{2}\frac{d}{du}\left(\frac{X^2}{1-M_p^2}\right) + \mu_0 r^2 \frac{dP_s}{du} + \frac{\mu_0}{2} r^4 \frac{d}{du}\left[\rho\left(\Phi'\right)^2\right] = 0 \qquad (1)$$

where $u(r,z) = \int_0^{\psi}\left[1-M_p^2(s)\right]^{1/2} ds$ with $2\pi\psi$ being the poloidal magnetic flux, M_p the poloidal Mach function, $X = X(u)$ a free function related to poloidal electric current, $\rho = \rho(u)$ the mass density function, $P_s = P_s(u)$ the static pressure and $\Phi = \Phi(u)$ the electrostatic potential. The GGS equation written in terms of the normalized quantities can be recovered by (1) just by setting $\mu_0 = 1$. A linear model which permits solutions with normal and reversed magnetic shear relies on the inclusion of a quadratic term in the functional dependence of the free function $\frac{X^2}{1-M_p^2}$ upon u. For example one should consider an immediate generalization of the Solovev lineariza-

tion as follows: $\frac{X^2}{1-M_p^2} = x_0 + 2x_1 u + x_2 u^2, \quad P_s = p_0 + p_1 u, \quad \rho\left(\Phi'\right)^2 = g_0 + 2g_1 u$. A similar model for static plasmas was studied recently in [3]. One may derive homogeneous solutions of eq. (1), employing the ansatz above, for both peaked and hollow current density profiles i.e. for $x_2 > 0$ and $x_2 < 0$

$$u_h(r,z) = \begin{cases} r\sum_j \left[a_j J_1\left(s^+ r\right) e^{jz} + b_j J_1\left(s^+ r\right) e^{-jz} + c_j Y_1\left(s^+ r\right) e^{jz} + d_j Y_1\left(s^+ r\right) e^{-jz}\right], & x_2 > 0 \\ r\sum_j \left[a_j J_1\left(jr\right) e^{s^- z} + b_j J_1\left(jr\right) e^{-s^- z} + c_j Y_1\left(jr\right) e^{s^- z} + d_j Y_1\left(jr\right) e^{-s^- z}\right], & x_2 < 0 \\ \textit{Solovev solution, e.g. see [4]}, & x_2 = 0 \end{cases}$$

where $s^+ = \sqrt{j^2 + x_2}$ and $s^- = \sqrt{j^2 - x_2}$ and J_1, Y_1 denote the first order Bessel functions of first and second kinds respectively. The complete solution is given by the superposition of a homogeneous solution with some particular solution of the inhomogeneous equation, that is $u = u_h + u_p$. One may find several particular solutions. Below we give a couple of them:

$$\begin{aligned} u_p^{(1)} &= -x_2^{-1}\left[x_1 + \left(p_1 - 8g_1/x_2\right) r^2 + g_1 r^4\right] \\ u_p^{(2)} &= -x_2^{-1}\left(x_1 + p_1 r^2\right) + \gamma_1 r J_1(\sqrt{x_2} r) + \gamma_2 r Y_1(\sqrt{x_2} r) + \\ &+ \frac{\pi g_1}{2x_2} r^4 \left[\sqrt{x_2} r J_4(\sqrt{x_2} r) - 4J_3(\sqrt{x_2} r)\right] Y_1(\sqrt{x_2} r) + \\ &+ \frac{\pi g_1}{4} J_1(\sqrt{x_2} r) r^6 G_{2,4}^{2,1}\left(\sqrt{x_2/4} r \Big|_{-1/2,1/2,-5/2,-1}^{-3/2,-1}\right) \end{aligned} \tag{2}$$

where G denotes the Meijer-G function and γ_1, γ_2 are arbitrary constants. D-Shaped and diverted Tokamak equilibrium configurations can be constructed (Fig. 1) by exploiting the free parameters (a_j, b_j, d_j, c_j) as described in the works [4]-[7]. The free parameters (p_1, x_1, x_2, g_1) are fixed in accordance with the experimental values of the various physical quantities and figures of merit, i.e. beta parameter $\beta \sim 1\%$, safety factor $q > 1$, $P \sim 10^5\ Pa$, $E \sim 10^4\ V/m$.

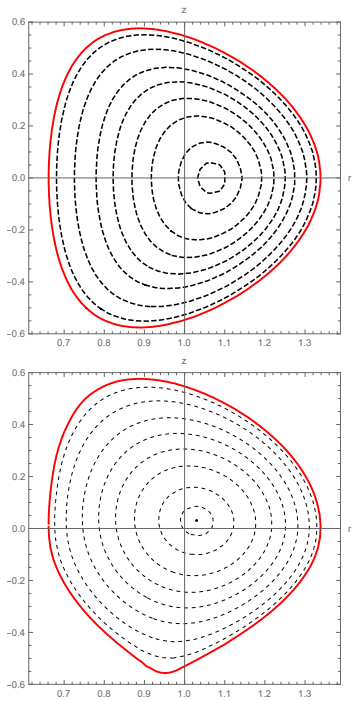

Figure 1: *Equilibrium configurations with smooth (up) and diverted (down) boundaries and ITER geometric characteristics.*

Equilibrium and magnetic transitions. We constructed equilibria with peaked and hollow current density profiles. Both kinds of equilibria with normal and reversed magnetic shear were observed. In this study the current column is shifted by imposing a condition for the position of the toroidal current density extremum. For typical Tokamak mass flows, the flow contribution is small enough in order to neglect its contribution in the imposition of this condition: $\partial_r\left[x_1/r + x_2 u(r,z)/r + p_1 r\right]\big|_{(r=1+\delta_r, z=\delta_z)} = \partial_z u(r,z)\big|_{(r=1+\delta_r, z=\delta_z)} = 0$, where $(1+\delta_r, \delta_z)$ is the position of the current density extremum. Along with certain boundary-shaping conditions [4]

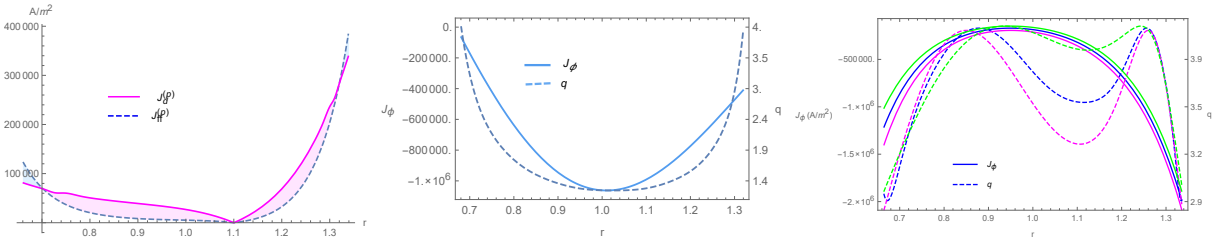

Figure 2: *(Left) The modulus of the poloidal diamagnetic current density versus the modulus of the poloidal force-free current density for a mixed diamagnetic-paramagnetic equilibrium (the diamagnetic plasma bulk is surrounded by a paramagnetic edge layer). (Center) The toroidal current density and safety factor profiles for a normal magnetic shear equilibrium with paramagnetic core and diamagnetic edge layer. (Right) The toroidal current compared to the safety factor for equilibria with hollow current densities and different triangularities. The shear reversals occur when the current density extremum is shifted slightly to the high-field side. The green, blue and magenta lines correspond to equilibria with triangularities* $\delta = 0.1$, $\delta = 0.3$, $\delta = 0.5$ *respectively.*

we derive equilibria with the desired shapes and the desired current density extremum position. It is interesting to note that for even very small values of δ_z the equilibria loose the desirable nested magnetic topology. This indicates that the vertical current displacements act in an unfavorable way as concerns the nestedness of magnetic surfaces. Varying the position δ_r we observed magnetic transitions which occur due to the boost of the poloidal component of the force free current density $\mathbf{J_{ff}} = \frac{\mathbf{J}\cdot\mathbf{B}}{B^2}\mathbf{B}$ versus the poloidal component of the diamagnetic current density $\mathbf{J_d} = \frac{\mathbf{B}\times\nabla P}{B^2}$. It has been stated [9] that the electric currents which flow parallel to the magnetic field can be driven by the helicity injections, hence we conjecture that one can possibly control a magnetic phase transition by adjusting the toroidal current density distribution, since it seems that it affects the force-free current density. We can briefly state that in both cases diamagnetism emerges when the Shafranov shift increases. The solution with $x_2 < 0$ results in core-diamagnetic equilibria with a paramagnetic outer layer when the Shafranov shift acquires typical values (Fig. 2). In view of a prototype transport model which was introduced by Solano and Hazeltine [10] we conclude that such an equilibrium is favorable for the reduction of radial transport. According to this model the high pressure diamagnetic plasma blobs are attracted by diamagnetic plasma regions while the low pressure paramagnetic blobs are attracted by paramagnetic regions. Therefore the diamagnetic core suppresses outward thermal transport due to blob convection. Also the external paramagnetic layer helps so as the plasma elements which have lost a fraction of their thermal energy due to some interaction with the wall, be constrained from interacting significantly with the inner hot plasma elements.

Stability Analysis. The linear stability analysis was conducted by employing a sufficient stability condition for constant mass density plasmas with incompressible flows parallel to the

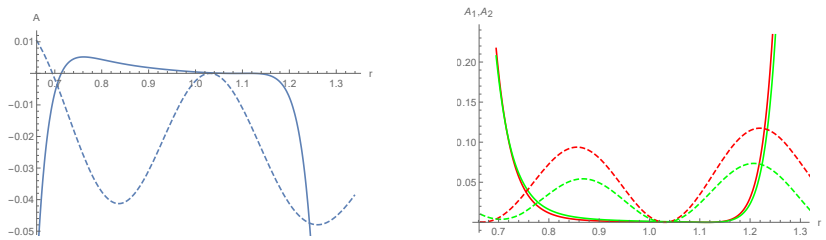

Figure 3: *(Left) The quantity A [Eq. (6)] on the equatorial plane for equilibrium with normal magnetic shear (dashed) and equilibrium with magnetic shear reversal (solid). For the equilibrium with shear reversal, the stability condition $A > 0$ is satisfied in a broad region extended from the plasma core to the high field side. For the normal shear equilibrium the stability condition is satisfied only within a thin layer at the outer high-field side. (Right) The term A_1 (red) versus the term A_2 (green) for normal magnetic shear equilibrium (dashed) and equilibrium with shear reversal (solid). The "stable" regions emerge when A_2 predominates A_1.*

magnetic field [8]. The equilibria examined here in terms of this condition, were obtained by the solutions of the previous sections setting $g_1 = 0$ and $\rho = const$. The condition states that if the flow is sub-Alfvenic, i.e. $M_p < 1$ and the quantity A below, is positive, then the equilibrium is linearly stable. The quantity A is given by

$$A = A_1 + A_2 + A_3 + A_4, \quad A_1 = -\left(\mathbf{J} \times \nabla u\right)^2, \quad A_2 = \left(\mathbf{J} \times \nabla u\right) \cdot \left(\nabla u \cdot \nabla\right) \mathbf{B}$$
$$A_3 = -\frac{1}{4} \frac{\left(M_p^2\right)'}{1 - M_p^2} |\nabla u|^2 \nabla u \cdot \nabla B^2, \quad A_4 = \frac{1}{2} \frac{\left(M_p^2\right)'}{\left(1 - M_p^2\right)^2} |\nabla u|^4 \left[P_s' - \left(M_p^2\right)' B^2/2\right] \tag{3}$$

A general conclusion is that the stable regions are determined by the competition of the terms A_1 and A_2. The term A_1 acts always in a destabilizing manner and is possibly related to current driven instabilities. In the case of equilibria with magnetic shear reversals the term A_2, which is related to the variation of the magnetic field perpendicular to the magnetic surfaces, is enhanced significantly against A_1 in a considerably large plasma region thus indicating a stabilizing effect of reversed magnetic shear, (see Fig. 3). The contribution of the flow dependent terms A_3 and A_4 to A is much weaker.

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