Stability of long rows of magnetic electron vortices

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I. In the present paper we develop the nonlinear theory of the magnetic electron drift (MED) modes. These magnetic fluctuations are drift-type waves excited in non-uniform plasma and characterized by a frequency range in-between the electron and ion plasma frequencies. The linear theory of the modes shows that the unstable motion is physically connected with the creation of finite electron fluid vorticity by the baroclinic vector, $\nabla n_0 \times \nabla T_e$ (n_0 is the background density and T_e the electron temperature) and the phase velocity of linear waves is confined to a certain interval . The electron inertia, which manifests itself in the electron vorticity, and the temperature perturbation, are then essential components in studies of these modes. The typical frequency of the motion is of order κv (v is the electron thermal velocity, κ^{-1} is the characteristic length of the background inhomogeneity). Phenomena occurring in such time scales are important as a source of different magnetic structures encountered in space plasma, as well as in a number of plasma devices.

To perform the analysis we derive the two-field nonlinear equations for the MED modes. The Hamiltonian structure of these equations is identified and used then to find a complete set of time-independent integrals of motion, including so-called Casimirs. As a next step, we examine the stationary solutions of model equations and show that infinitely long rows of vortices, vortex chain or vortex streets are allowed by these nonlinear equations. In recent years, it has been appreciated that these solutions represent maximum entropy states which are believed to be the most probable final state in decaying 2-D Navier-Stokes and similar drift plasma turbulence. Finally, we proceed to the stability analysis. Lyapunov's direct method is used to investigate the stability of stationary solutions with respect to small perturbations (linear analysis). On particular example of vortex streets the linear stability for long wave length perturbations is established. It is also shown that nonlinear stability cannot be proven using Arnold's method.

II. The motion of the considered modes is assumed to take place in the plane perpendicular to the magnetic field and hence a quasi-two-dimensional analysis is applied, where only the perturbed magnetic field is directed along the third dimension, here chosen to be the z axis. These modes are placed in a non-uniform unmagnetized plasma with density and temperature gradients along the x axis. The temperature and density gradients of the fluctuations are in general not collinear, and this generates a vorticity in electron fluid. The consequent motion generates a perpendicular magnetic field (with vanishing equilibrium part), $B(x, y, t)\mathbf{z}$, which actually plays the role of a stream function. Due to a typical time scale of the MED modes, the ions play the role of a neutralizing background in the mode dynamics, whereas the electrons move fast enough to equalize any density perturbation in a relatively short time. Therefore, the electron density will be considered constant on time, $n = n_0$. The temperature can be written as the sum of an equilibrium value T_0 and a perturbation T. We assume that the length scale of the fluctuations is much smaller than that of the equilibrium one (this can be expressed by small parameters $\varepsilon_n \sim |\nabla \ln n_0|/k$ and $\varepsilon_T \sim |\nabla \ln T_0|/k$, where $k^{-1} \sim c/\omega_{pe}$ is the typical spatial scale of the fluctuations), and take $\varepsilon_n \sim \varepsilon_T \sim \varepsilon$. Starting then from the momentum equation and the energy equation, the model equations for MED mode turbulence can be derived up to the lowest non-vanishing order in ε and read in dimensionless form

$$\frac{\partial}{\partial t}\left(B-\nabla^2 B\right)-\left\{B,\nabla^2 B\right\}=-v_0\frac{\partial T}{\partial y} \tag{1a}$$

$$\frac{\partial T}{\partial t}+\left\{B,T\right\}=-w_0\frac{\partial B}{\partial y} \tag{1b}$$

Here, $v_0=\left|\nabla \ln n_0(x)\right|$, $w_0=T_0\left(2v_0/3-\left|\nabla \ln T_0\right|\right)$ may be regarded as constant coefficients, the length unit is $\left(c/\omega_{pe}\right)$, the magnetic field and the temperature are normalized by $\left(e/m\right)B \to B$, and $\left(\omega_{pe}^2/c^2 m\right)T \to T$, the curl brackets denote the Poisson brackets and are defined as $\left\{a,b\right\}\equiv\left(\nabla a\times\nabla b\right)\cdot\mathbf{z}$. The dispersion relation of the linear version of Eqs.(1) is

$$\omega^2=v_0 w_0\left[k_y^2/\left(1+k^2\right)\right] \tag{2}$$

Note that a purely growing solution is possible for $\varepsilon_T > 2/3\varepsilon_n$, or $\left(v_0 w_0\right)<0$, which can explain the measured strong magnetic fields in laser-produced laser experiments. Of course, due to this linear growth, the linear approximation breaks down and nonlinear effects have to be included. On the other hand, in a stable plasma, $\left(v_0 w_0\right)>0$, the phase velocity of linear waves in the y direction has an upper limit, indeed, $-\left(v_0 w_0\right)^{1/2}<\omega/k_y<\left(v_0 w_0\right)^{1/2}$.

III. We now consider some general properties of Eqs.(1). Introducing variables $q=B-\nabla^2 B$ and $\phi=T-w_0 x$, yields the energy integral in the form

$$E=0,5\int\left(B^2+\left|\nabla B\right|^2+\frac{v_0}{w_0}T^2\right)\mathrm{dxdy}\to\; H=0,5\int\left(Bq+\frac{v_0(\phi+w_0 x)^2}{w_0}\right)dxdy \tag{3}$$

If now we introduce the state vector $\mathbf{u}=\begin{pmatrix} q \\ \phi \end{pmatrix}$, then it easy to see that Eqs.(1) can be presented in the Hamiltonian form

$$\frac{\partial \mathbf{u}}{\partial t}=J\cdot\frac{\delta H}{\delta \mathbf{u}},\quad \text{where } J=\begin{pmatrix} \{q,\cdot\} & \{\phi,\cdot\} \\ \{\phi,\cdot\} & 0 \end{pmatrix}$$

is the noncanonical Poisson matrix, $\delta/\delta\mathbf{u}$ is the usual variational derivative. The functional H naturally plays the role of Hamiltonian. We employ the Hamiltonian structure to find the set of integrals, using general methods for noncanonical systems. To this end we form the Lie-Poisson bracket $\left[F,H\right]$ with

$$\left[F,H\right]=\int\frac{\delta F}{\delta\mathbf{u}}\cdot\left(J\cdot\frac{\delta H}{\delta\mathbf{u}}\right)dxdy=\int\frac{\delta F}{\delta\mathbf{u}}\cdot\frac{\partial\mathbf{u}}{\partial t}dxdy=\frac{dF}{dt}$$

if $\partial F/\partial t=0$. Hence, if $\left[F,H\right]=0$, F is an integral of the system. We use this to construct so-called Casimir invariants, which satisfy $J\cdot\left(\delta C/\delta\mathbf{u}\right)=0$. The only two solutions are

$$C_f=\int qf\left(\phi\right)dxdy \quad \text{and} \quad C_g=\int g\left(\phi\right)dxdy \tag{4}$$

Furthermore, since the Hamiltonian contains a continuous symmetry in the y direction we can find the remaining integral of the basic equations using Noether's theorem. This integral is written as

$$M=\int xqdxdy \tag{5}$$

and can be interpreted as a conserved momentum. We have thus identified the Hamiltonian structure of Eqs.(1) and obtained all time-independent integrals.

IY. As a next step, we examine the stationary solutions of Eqs.(1) which propagate with constant velocity $u\hat{\mathbf{y}}$. Setting $\partial/\partial t=-u\,\partial/\partial y$ and introducing the stream function $\psi=B-ux$ we find that the stationary solution will be given by

$$\nabla^2\psi=r\left(\psi\right)+ux\left(1-\frac{v_0}{u}s'\left(\psi\right)\right) \quad \text{and} \quad T=s\left(\psi\right)+w_0x \tag{6}$$

where r and s are arbitrary functions. We show now that in the set defined by (6) there exist stationary solutions which are localized in one direction and periodic in the other. To this end we choose $s\left(\psi\right)=\dfrac{u}{v_0}\psi$, so that the first expression in the set (6) is reduced to $\nabla^2\psi=r\left(\psi\right)$, which is the relation between the stream function $\left(\psi\right)$ and the vorticity often used in the fluid dynamics. Consider two possible particular cases, namely,

$$r_1\left(\psi\right)=\xi\sinh\psi \quad \text{and} \quad r_2\left(\psi\right)=A\exp\left(-\psi/A\right) \tag{7a}$$

which correspond to the "sinh-Poisson" equation and to the Liouville-equation. The solutions of these equations are well-defined in 2-D fluid dynamics and under some restrictions on free parameters they describe, physically, so-called "vortex streets". If $\nabla^2\psi=r_{1,2}\left(\psi\right)$, the solutions to these equations are known as the "breather", ψ_1, and Kelvin-Stuart cat's eyes, ψ_2, and are given by

$$\psi_1=4\operatorname{arc\,tanh}\left(\sqrt{\frac{b}{a}}\frac{\sin\sqrt{a}\,y}{\cosh\sqrt{b}x}\right), \text{ and } \psi_2=2A\ln\left[8b^2\left(2a\cosh bx+2\sqrt{a^2-1}\cos by\right)\right] \tag{7b}$$

where $\xi=b-a,\ a>0,\ b>0\ \left(\psi_1\right)$ and $a>1\ \left(\psi_2\right)$, a and b are arbitrary constants. As can be seen from (7), these solutions describe vortex flow $\left(\nabla^2\psi\neq0\right)$ which is localized in the x direction and periodic in y. In the Kelvin-Stuart cat's eyes solution, ψ_2, the parameter a describes the width of the cat's eyes. As a decreases to 1 the cat's eyes diminish and the limiting flow is purely zonal.

Y. We now proceed to the stability analysis of stationary solutions given by Eqs.(7) using the Lyapunov's direct method. To this end, we construct a Lyapunov functional L with zero first variation for (7), by means of integrals (3-5), $L = E + C_f + C_g + \lambda M$, where λ is a Lagrange multiplier. The first variation of L is zero for solutions (7) if $g'(\phi) = -\alpha^2\left(\nabla^2\phi - \phi\right) - \frac{v_0}{w_0}\phi$, here we assume $f(\phi) = \alpha\phi$ and chose $\alpha = v_0/\lambda$ and $\lambda = -u$. Now, inserting into $g'(\phi)$ stationary solutions (7) results in the following expression for $\delta^2 L$:

$$\delta^2 L = \int\left\{\left(\delta B + \alpha\delta T\right)^2 + \left|\nabla\delta B + \alpha\nabla\delta T\right|^2 - \alpha^2\left[\left|\nabla\delta T\right|^2 + r'\left(\frac{v_0}{u}\phi\right)\left(\delta T\right)^2\right]\right\}dxdy \qquad (8)$$

To ensure stability the second variation of L should be of definite sign. If $r' > 0$ nothing can be concluded about stability from (8) unless a specific relation between δB and δT is assumed. On the other hand, if $r' < 0$ we can estimate upper bound on the linear perturbation wave number, k_0, for which we have stability. To this end we need the lower bound of $|r'|$. We actually have, for solutions (7a) and (7b),

$$0 < |\zeta| \le |r_1'| \le |\zeta| + \frac{8ab}{|\zeta|} \le \infty, \quad 0 < 8b^2\left(2a\cosh bx_{\max} + 2d\right)^{-2} \le \left|r_2'\right| \le 8b^2\left(2a - 2d\right)^{-2} < \infty \quad (9)$$

where $d = \sqrt{a^2 - 1}$ and the region of consideration is limited by $\pm x_{\max}$.Then the stability criterion $\delta^2 L > 0$ takes the form $k_0^2 < c_1$, where c_1 is the lower limit of $|r_1'|$ or $|r_2'|$ in (9). So, the linear stability of the stationary solutions (7) for long wavelength perturbations is proved. Consider, for example the geometry of the solution ψ_1. If the scale of a single vortex along the direction of the chain is Δy, and the transverse scale is Δx, then stability condition can be roughly stated $k_0^2 + (\Delta x)^{-2} \le (\Delta y)^{-2}$. Hence the solution is linearly stable to perturbations of scale larger than Δy, and the wider the vortices (in the x direction) the shorter the scale of the perturbations may be up to this limit.

YI. To study the nonlinear stability properties of our stationary solutions (7) we employ the method introduced by Arnol'd for two dimensional incompressible flow. According to this method, the nonlinear generalization of $\delta^2 L$, defined as $\tilde{L} = L\left(B_0 + B_1, T_0 + T_1\right) - L\left(B_0, T_0\right)$, where B_0 and T_0 are the stationary solutions and B_1 and T_1 are small but finite perturbations, should be restricted from above and below by positive definite quadratic forms. Then, it can be shown, following the Arnol'd method, that an initial finite perturbations will remain bounded for all time. Direct applications of this method to our system of equations, when B and T vary arbitrarily and independently shows that no definite inference could be made in the general. We believe that the stability analysis by the above mentioned method is in need of complementary ideas. Numerical evidence that such structures are stable suggest that even if the vortices are not stable to arbitrary large or global perturbations, they should at least be expected to be stable to finite disturbances.