# Translational symmetry with phase shift in Ballooning Mode

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## Introduction

Understanding of two-dimensional structure of the toroidal drift waves is a long standing subject in fusion research. I revisited this issue recently for writing an introductory book on modern tokamak physics [1]. Lee-Van Dam[2] and Zakharov [3] assumed a radial translational symmetry to derive a quasi-mode representation of the ballooning mode, while J. Connor derived it based on the Fourier transformation [4]. Here I show a smooth way to connect these two approaches including a phase shift $\theta_k$ by use of $\delta$-function formula.

Two dimensional eigenmode structure is categorized into passing and trapped modes[5] and the formula for radially overlaped mode envelope widths are discussed by Romanelli[8] and Taylor[10] for the trapped mode and by Kim-Kishimoto[11] for the passing mode. I revisited these formula and slightly modified result is obtained.

## Poloidal harmonics expression from the Eikonal form

In the ballooning approximation $k_\parallel \ll k_\perp$ in an axisymmetric tokamak, the eigenmode $\varphi(r,\theta,\zeta)$ to satisfy double periodicity in $(\theta,\zeta)$ is given by the infinite summation of the Eikonal form $\hat{\varphi}(r,\eta,\zeta) = u(r,\eta)e^{iS(r,\alpha)}$ in the covering space $\eta \in (-\infty,+\infty)$ using $S = -n(\alpha+\alpha_0(r))$ as,

$$\varphi(r,\theta,\zeta) = e^{-in\zeta}\sum_{j=-\infty}^{+\infty} u(r,\theta+2\pi j)e^{inq(\theta-\theta_0+2\pi j)}, \text{ where } \quad q(r)\theta_0(r) \equiv \int \theta_k dq \qquad (1)$$

Here the coordinates $(r,\theta,\zeta)$ is a flux coordinates and $\theta_k(r) = \alpha_0'(r)/q'(r)$[5], [6], $\alpha = \zeta - q\theta$. Using the $\delta$ function, an integral form is obtained.

$$\varphi(r,\theta,\zeta) = e^{-in\zeta}\sum_{j=-\infty}^{+\infty}\int_{-\infty}^{+\infty} d\eta u(r,\eta+\theta_0)\delta(\eta-(\theta-\theta_0+2\pi j))e^{inq\eta} \qquad (2)$$

If we apply the delta function formula $2\pi\sum_{j=-\infty}^{+\infty}\delta(x-2\pi j) = \sum_{m=-\infty}^{+\infty} e^{-imx}$ and set $x = \eta-\theta+\theta_0$, we have:

$$\varphi(r,\theta,\zeta) = e^{-in\zeta}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{d\eta}{2\pi}u(r,\eta+\theta_0)e^{i(nq-m)\eta}e^{im(\theta-\theta_0)} \qquad (3)$$

If we define $\eta' = \eta+\theta_0$, above equation is rewritten as,

$$\varphi(r,\theta,\zeta) = e^{-in\zeta}\sum_{m=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{d\eta'}{2\pi}u(r,\eta')e^{i(nq-m)\eta'}e^{-inq\theta_0}e^{im\theta} \qquad (4)$$

In this expression, we see $nq-m$ is a fast varying radial variable conjugate to $\eta'$. Renaming $\eta'$ to $\eta$, we have following expression for the eigen function.

$$\varphi(r,\theta,\zeta) = e^{-in\zeta} \sum_{m=-\infty}^{+\infty} \varphi_0(r,nq-m) e^{-inq\theta_0} e^{im\theta} \tag{5}$$

Here, $\varphi_0(r,nq-m)$ is given by the following radial Fourier integral.

$$\varphi_0(r,nq-m) = \int_{-\infty}^{+\infty} \frac{d\eta}{2\pi} u(r,\eta) e^{i(nq-m)\eta} \tag{6}$$

We find that the phase shift term is not $e^{-im\theta_0}$ but is $e^{-inq\theta_0} = e^{-i\int_{q_0}^{q}\theta_k d(nq)}$.

**Bloch Angle and Radial Envelope Formula**

The function $\varphi(r,nq-m)$ represents a localized eigenfunction near the rational surface $q(r_m) = m/n$ having the translational symmetry in the radial direction $--, r_{m-1}, r_m, r_{m+1}, --$. If we pick up phase shifts for these rational surfaces ($\Delta nq = 1$), we have,

$$\int_{nq_0}^{nq} \theta_k(q) d(nq) \sim \sum_{j=nq_0}^{m} \theta_k(j/n) \tag{7}$$

The phase shift between adjacent rational surface is called the "Bloch angle" due to its similarity to the solid state physics but changes radially. The Bloch angle is therefore $\theta_k$ and not $\theta_0$ defined in (1) adopted from Hazeltine-Meiss [14]. If the $\varphi_0(r,nq-m)$ is extremely peaked at $nq = m$, we can approximate $e^{-inq\theta_0} \to e^{-im\theta_0}$ to reach,

$$\varphi(r,\theta,\zeta) = e^{-in\zeta} \sum_{m=-\infty}^{+\infty} \varphi_0(r,nq-m) e^{im(\theta-\theta_0)} \tag{8}$$

In this case, the phase angles of all poloidal harmonics becomes zero at $\theta = \theta_0$ so that we can argue $\theta_0$ is a measure of poloidal angle where the mode elongation is purely radial as discussed by Kishimoto [12] but may not be the tilting angle [13].

If we expand $\theta_k(r)$ in Taylor series as $\theta_k(r) = \theta_k(r_m) + \theta_k'(r_m)(r-r_m) + --$, we can assume $\theta_k(r_m)$ is real if the mode amplitude is maximum at $r = r_m$. However, the $\theta_k'(r_m)$ may have both real and imaginary parts. For the imaginary part $Im[\theta_k]$, the phase shift term $e^{-i\int\theta_k d(nq)}$ is,

$$exp\left[n\int Im[\theta_k]dq\right] = exp\left[-\alpha(r-r_m)^2\right], \text{ where } \alpha = -Im[\theta_k'(r_m)]nq'(r_m)/2 \tag{9}$$

This $\alpha$ characterizes the radial full width $\Delta r = 2\alpha^{-1/2}$ of ballooning mode peaked at $r = r_m$. For the real part $Re[\theta_k]$, the phase shift term $e^{-in\int Re[\theta_k]dq}$ is expanded as,

$$exp\left[-i[\theta_k(r_m)nq'(r_m)(r-r_m) + Re(\theta_k'(r_m))nq'(r_m)\frac{(r-r_m)^2}{2} + --]\right] \tag{10}$$

So the $Re(\theta_k'(r_m))$ is related to poloidal tilting of radially elongated mode structure.

**Local Dispersion Relation**

The local dispersion relation may be given as $F(\omega, q, \theta_k) = 0$, where $q$ is a radial coordinate. Dewar[6] analyzed characteristics of dispersion relation and applied Bohr-Sommerfeld condition for trapped and passing modes in general two dimensional eigenmode equation in particular for the ideal MHD stability. Later, Taylor[9] discussed Bohr-Sommerfeld condition for application to toroidal drift waves. Zonca-Chen [7] applied this technique in TAE analysis (trapped mode) and Romanelli-Zonca [8] for ITG mode. Number of authors calculated the local dispersion relations.

**Envelope width for Passing Mode**

If the local dispersion function $F(\omega, q, \theta_k)$ is monotonic ($\partial F/\partial q \neq 0$, $\partial F/\partial \theta_k \neq 0$), we write the dispersion relation by $\omega = \omega(r, \theta_k) = \omega_r(r, \theta_k) + i\omega_i(r, \theta_k)$ and the mode is passing mode (or not trapped) in a sense of Dewar [5], the radial derivative of the local dispersion relation is given by $\partial_r \omega + (\partial_{\theta_k} \omega)\theta_k'(r_m) = 0$, which gives following expression for $\alpha$.

$$\alpha = Im \left[ \frac{\partial_r \omega}{\partial_{\theta_k} \omega} \right] \frac{nq'(r_m)}{2} = -\frac{\partial_r \omega_r}{\partial_{\theta_k} \omega_i} \frac{nq'(r_m)}{2} \tag{11}$$

Here, the last equation is derived assuming that the mode frequency $\omega$ is dominated by the real oscillation so that $\partial \omega/\partial r$ is also dominated by the real part. Addition of the Doppler-shifted real frequency and a particular choice of growth rate $\omega_i = \gamma_0 cos\theta_k$ leads to Kim-Kishimoto formula for the full width of radially overlapped envelope[13] using $nq' = k_\theta s$ where $s = rq'/q$ is the magnetic shear.

$$\Delta r = 2\sqrt{\frac{2\gamma_0 sin\theta_k}{k_\theta s \partial_r(\omega_r + \omega_f)}} \tag{12}$$

**Envelope width for Trapped Mode**

If the local dispersion function $F(\omega, q, \theta_k)$ has extremal at $(q, \theta_k) = (q_0, \theta_{k0})$, radial mode width is given in a different form. We may expand in a quadratic form using $F_0 = -F(\omega, q_0, \theta_{k0})$.

$$-F_0 + \frac{1}{2}\left[F_{qq}(q - q_0)^2 + F_{\theta_k\theta_k}(\theta_k - \theta_{k0})^2\right] = 0 \tag{13}$$

Here $F_{qq} = \partial^2 F/\partial q^2$ and $F_{\theta_k\theta_k} = \partial^2 F/\partial \theta_k^2$. The equi-contour surface becomes elliptic and gives two branches of solutions $\theta_k^\pm(\omega, q) = \theta_{k0} \pm \sqrt{F_{qq}/F_{\theta_k\theta_k}}\sqrt{a^2 - (q - q_0)^2}$ to satisfy the local dispersion relation. Here $a^2 = 2F_0/F_{qq} = (q_2 - q_1)^2/4$. The WKBJ solutions are given by $exp(-in\int^q \theta_k^\pm dq)$. In this case, right and left travelling waves produces a standing wave if the Bohr-Sommerfeld condition $n\int_{q_1}^{q_2}(\theta_k^+ - \theta_k^-)dq = 2\pi(N + 1/2)$ is met. Changing integration from $dq$ to $d\varphi$ by the transformation $q - q_0 = asin\varphi$ and let $N = 0$, we have,

$$q_2 - q_1 = \frac{2}{\sqrt{n}}\left(\frac{F_{\theta_k\theta_k}}{F_{qq}}\right)^{1/4} \tag{14}$$

We obtain slightly different formula compared with the equation (11) in [10].

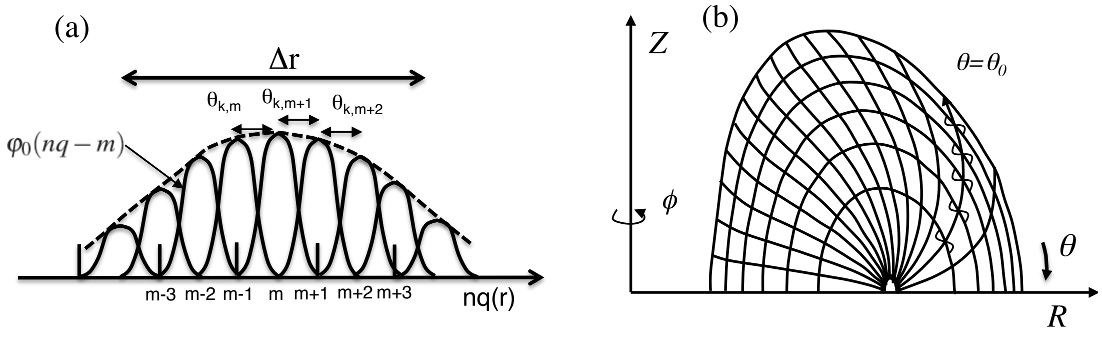


Figure 1: (a) Radially overlapped ballooning eigenmode structure with phase shift $\theta_k$ and $1/e$ full width of envelope. (b) Example of phase alignment in the flux coordinates with constant $\theta_0$. In the flux coordinates, constant poloidal angle line is curved in $(R,Z)$ plane and tilting due to $\theta_k'$ will be superposed on it.

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