An efficient method of solution for collisional relaxation of trapped particles

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A potential well containing trapped particles as well as dot sources and/or sinks for these particles is considered. The kinetic boundary-value problem to be solved is to find the stationary velocity distribution function (VDF) of the trapped particles, assuming the presence of a relaxation process driving any non-equilibrium VDF towards the equilibrium one. This problem is solved on the basis of the authors' previous expertise on a variety of pertinent kinetic boundary-value problems [1-3].

The general goal of this work and recent similar efforts by the authors [4] is to introduce a new, efficient methodology of solving kinetic boundary–value problems with localized particle sources and sinks. The novelty and efficiency of the method can be summarized as follows: (i) The relevant VDFs are calculated via trajectory integration of the underlying kinetic equations. This allows for (ii) inclusion of boundary conditions in a natural way and (iii) the possibility of obtaining piecewise analytic solutions in collisionless regions. (iv) The latter can be matched via the interface conditions to yield the global VDF.

In free space, the particles' VDF $h(r,v)$ is assumed to satisfy the collisionless kinetic equation (Vlasov equation)

$$\frac{Dh}{Dt} = v \cdot \frac{\partial h}{\partial r} - \frac{dU}{dr} \cdot \frac{\partial h}{\partial v} = 0 . \qquad (1)$$

The entrance boundary conditions at the left- and right-hand boundaries are prescribed as

$$h\left(R_{1,}, v > 0\right) = g_1\left(\frac{v^2}{2}\right) \quad \text{and} \quad h\left(R_2, v < 0\right) = g_2\left(\frac{v^2}{2}\right), \qquad (2)$$

respectively, where $r \in [R_1, R_2]$ is position, $v \in (-\infty, +\infty)$ is velocity, R_1 and R_2 are the left- and right-hand boundaries ("electrodes") of the region considered, g_1 and g_2 are any non-negative functions, and the "potential energy" U is defined as $U(r) = ZeV(r)/m$, with $V(r)$ the electrostatic potential, e the positive elementary charge, and m and Z the single-particle mass and charge number, respectively. As is well known, the collisionless single-

particle trajectories are curves of constant total particle energy, $W(r,v)=\frac{1}{2}v^2+U(r)=\text{const}$, the "Lagrangian" time derivative $\frac{D}{Dt}=v\cdot\frac{\partial}{\partial r}-\frac{dU}{dr}\cdot\frac{\partial}{\partial v}$ is the rate of change as "seen" by a particle moving along its collisionless trajectory, and the Vlasov equation (1) states that the VDF h is constant along any collisionless trajectory.

Hence, the collisionless trajectories generally play a crucial role in formulating and solving kinetic boundary-value problems, so that it is extremely important and helpful to familiarize oneself with their topology in (r,v) phase space for any particular case. Here we point out that the "separating trajectory" ("separatrix") between trapped- and untrapped-(transiting-) particle trajectories corresponds to the energy $W_s(r)=U^*$, hence has the form $v_s(r)=\pm|v_s|(r)=\pm\sqrt{2\left[U^*-U(r)\right]}=v_s^{\pm}(r)$ and clearly exhibits an upper (+) and a lower (-) branch. The phase-space regions above the upper and below the lower branch are occupied by the trajectories entering at R_1 and R_2, respectively ($W(r,v)>U^*$), while the region between the two branches is occupied by the trapped trajectories/particles ($W(r,v)<U^*$). Let us denote the entrance velocities of trajectories entering at $R_{1,2}$ by $w_{1,2}$, respectively, with $w_1>0$ and $w_2<0$. Upon proper combination of these elements we find, in the absence of sources, sinks and trapped particles, the general solution $h(r,v)$ of the kinetic boundary-value problem (1), (2) in the form

$$h=H_+\cdot g_1\left(\frac{1}{2}w_1^2\right)+H_-g_1\left(\frac{1}{2}w_2^2\right), \tag{3}$$

where the Heaviside unit-step functions $H_+(r,v)=H\left(v-|v_s|(r)\right)$ and $H_-(r,v)=H\left(-v-|v_s|(r)\right)$ pick out the phase-space regions above the upper and below the lower separatrix branch, respectively, $w_{1,2}(r,v)=\pm\sqrt{v^2+2\left[U(r)-U^*\right]}$, and $\frac{1}{2}w_{1,2}^2=\frac{1}{2}v^2+U(r)-U^*$. For later use, we here also introduce the Heaviside function picking out the phase-space region for trapped trajectories/particles, namely $H_{tr}(r,v)=H\left(|v_s|(r)-|v|\right)=H\left(U^*-W(\mathrm{r},v)\right)\quad=H\left(U^*-U(r,v)-\frac{1}{2}v^2\right)$.

In the subsequent "stages" of our work we will insert into the "diode" configuration just considered dot sources and sinks of various kinds, so that we are bound to get faced with kinetic boundary-value problems of increasing complexity and we are well advised to choose

our way of proceeding quite judiciously. To be specific, we choose to proceed as follows: While situations of significantly higher complexity can be tackled in the future, we here restrict ourselves, as is appropriate for the present "first stage", to the following, relatively simple exemplary problem for which the physics is obvious.

Let us consider a potential well in which there is only a dot source (Fig. 1) or only a dot sink (Fig. 2) for the particles. It is obvious that in the first case, where particles only enter the potential well but do not leave it, the VDF should tend to infinity, while in the second case, when particles are only removed but are not replenished, the VDF should tend to zero.

In Figs. 1 and 2, the dot source and sink are placed at $r = t$. The "source VDF" indicated in Fig. 1 is chosen as $g_t = g\left(\frac{1}{2}v^2, t\right) \cdot H_{tr}(t, v)$, which because of the function H_{tr} means that the *source emits only trapped particles*. In Fig. 2, on the other hand, it is assumed that on passage of the particles through the dot sink the VDF $h(r, v)$ is modified according to $h \to \alpha \cdot h$, with the "sink factor" $\alpha < 1$.

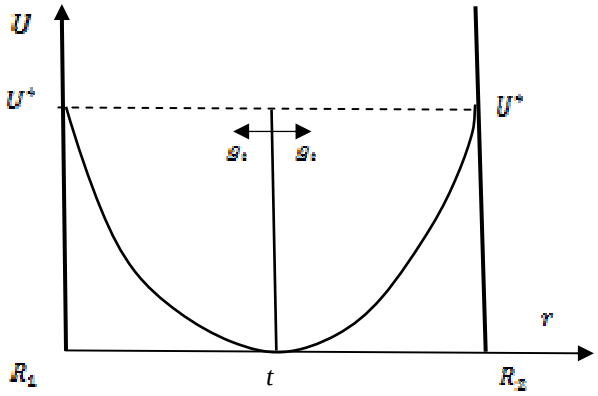

Fig. 1

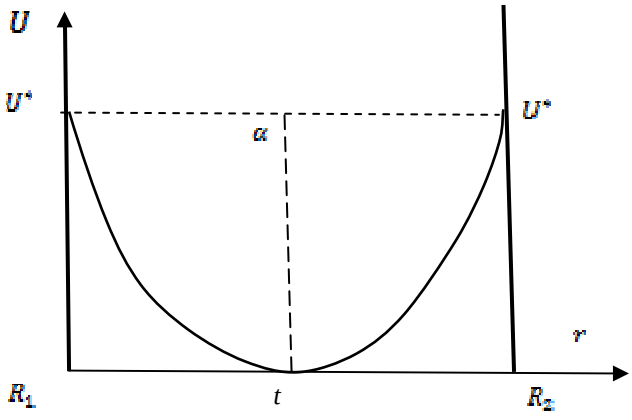

Fig. 2

With the source present at $r = t$ (Fig. 1) there are two regions in each of which we can formulate a kinetic boundary-value problem for the Vlasov equation, namely Region 1 ($r \in [R_1, t]$) and Region 2 ($r \in [t, R_2]$), for which we choose the boundary conditions

$$h_1(R_1, v > 0) = 0, \qquad h_1(\mathrm{t}, v < 0) = \left[g_t + h_2(t)\right] \cdot H(-v), \tag{4a}$$

$$h_2(\mathrm{t}, v > 0) = \left[g_t + h_1(t)\right] \cdot H(v), \qquad h_2(R_2, v < 0) = 0. \tag{4b}$$

Let us solve this problem by means of an iterative process as follows. In the ***1st iteration step,*** we assume that Regions 1 and 2 are isolated from each other. Then the boundary conditions (4a,b) at the point $r = t$ assume the form $h_{11}^{(1)}(\mathrm{t}, v < 0) = g_t \cdot H(-v)$ and $h_2^{(1)}(\mathrm{t}, v > 0) = g_t \cdot H(v)$, respectively. The solution in Region 1 (see Fig. 1 and expressions (3)) then reads

$h_1^{(1)} = H(-\lambda_1)\cdot g(W-U_t,t)\cdot H(U^*-W)$ which after some intermediate calculations finally yields $h_1^{(1)} = g(W-U_t,t)\cdot H(U^*-W)$. Similarly, starting out from (4b) the solution of the collisionless kinetic boundary-value problem for Region 2 (see Fig. 1. and expressions (3)) is found as $h_2^{(1)} = g(W-U_t,t)\cdot H(U^*-W)$.

In the ***2nd iteration step*** and the ***subsequent iteration steps*** it is assumed that particles can freely pass from Region 1 into Region 2 and vice versa. Hence, taking into account the expressions for $h_1^{(1)}$ and $h_2^{(1)}$, the boundary conditions (4a,b) at $r = t$ become $h_1^{(2)}(R_1, v>0) = 0$, $h_1^{(2)}(\mathrm{t}, v<0) = 2\cdot g_t \cdot H(-v)$ and $h_2^{(2)}(\mathrm{t}, v>0) = 2\cdot g_t \cdot H(v)$, $h_2^{(2)}(R_2, v<0) = 0$, respectively. Calculations similar to those of the 1st iteration step lead to the following unified forms of the VDFs in Regions 1 and 2: $h_1^{(2)} = 2\cdot g(W-U_t,t)\cdot H(U^*-W)$.

Our further calculations performed under this scheme have shown that for the *k*-th iteration step the expressions found for the functions $h_1^{(k)}$ and $h_2^{(k)}$ are of the form $h_1^{(k)} = k\cdot g(W-U_t,t)\cdot H(U^*-W)$ and $h_2^{(k)} = k\cdot g(W-U_t,t)\cdot H(U^*-W)$, respectively, which confirms our expectation that for $k\to\infty$ both $h_1^{(k)}\to\infty$ and $h_2^{(k)}\to\infty$.

In the ***sink*** case (Fig. 2) it is assumed that the entire potential well is "filled" with trapped particles with the VDF $G(r,v) = g(W-U_t,t)\cdot H(U^*-W)$. Calculations similar to those for the source case have shown that for the *k*-th iteration step the functions $h_1^{(k)}$ and $h_2^{(k)}$ are of the form $h_1^{(k)} = \alpha^k \cdot g(W-U_t,t)\cdot H(U^*-W)$ and $h_2^{(k)} = \alpha^k \cdot g(W-U_t,t)\cdot H(U^*-W)$, respectively, so that because of $\alpha<1$ both tend to zero as $k\to\infty$, which is in line with our expectations.

Conclusion. In view of the fact that in both the source and the sink case the asymptotic results are obviously correct, we conclude that our new method of solving kinetic boundary-value problems with sources and sinks has passed this "test" successfully.

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