

Homoclinic tangle of the primary separatrix in the compact and closed magnetic topology for divertor tokamaks

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ABSTRACT: The equilibrium field lines Hamiltonian for the simple map [1] is modified by replacing the cubic canonical momentum term to a quartic. This results in changing the topology of the separatrix from open and unbounded to closed and compact. The new map is the symmetric quartic map (SQM) [2]. Parameters in the generating function of the SQM are chosen such that the height, width, elongation, and the poloidal flux inside the separatrix for the SQM are same as in the simple map. The map parameter k of the SQM is used to represent the magnetic perturbation as in the Standard Map [3]. The homoclinic tangle of the separatrix is calculated for different values of the map parameter using the forward and the backward symplectic maps. The purpose is to investigate what role the topology of the separatrix plays in its homoclinic tangle in single-null divertor tokamaks. This work is supported by grants DE-FG02-01ER54624, DE-FG02-04ER54793. This research used resources of the NERSC, supported by the Office of Science, US DOE, under Contract No. DE-AC02-05CH11231.

Magnetic field lines are the trajectories of a $1\frac{1}{2}$ degree of freedom Hamiltonians. Plasmas in tokamaks are confined in regions where the magnetic field lines form closed toroidal surfaces. These surfaces are bounded by a separatrix, and outside the separatrix the magnetic field lines and the plasma flow to special regions of the walls called divertors. Both the confinement of the plasma and the feasibility of divertors are sensitive to the behavior of the magnetic field lines near the separatrix in the presence of non-axisymmetric magnetic perturbations. Separatrix manifold forms homoclinic tangle to preserve the symplectic invariant and topological neighborhood as the manifold evolves in canonical time. At a homoclinic point, four manifolds join together; the two incoming stable separatrix manifolds, and the two outgoing unstable separatrix manifolds. The stable manifold M^S leading to and the unstable manifold M^U emanating from the hyperbolic point have extremely irregular behavior. This is because these two manifolds cannot intersect themselves but the unstable manifold M^U can intersect the stable manifold M^S at *homoclinic points*. Between each homoclinic point and the hyperbolic fixed point, there are an infinite number of homoclinic points. Thus the stable manifold M^S and the unstable manifold M^U form an extremely complex network called the *homoclinic tangle* [4,5]. For manifolds connected to neighboring hyperbolic points, these

structures are called *heteroclinic tangle*. In equilibrium, the axisymmetric separatrix manifold is degenerate, and the stable and unstable manifolds coincide. The magnetic field in toroidal confinement schemes can be expressed as $\vec{B} = \nabla\psi \times \nabla\theta + \nabla\varphi \times \nabla\chi(\theta, \psi, \varphi)$ [2,6]

where χ is a poloidal magnetic flux contained inside a magnetic surface and it plays the role of Hamiltonian, θ is a poloidal angle and plays the role of generalized coordinate, ψ is a toroidal magnetic flux inside a surface and plays the role of generalized momentum, and φ is a toroidal angle and plays the role of generalized time. There are three sets of canonical coordinates for magnetic field line trajectories in tokamaks; magnetic, natural, and physical. Physical coordinates are (x, y, φ) where $x = r \cos(\theta)$, $y = r \sin(\theta)$ for $\varphi = \text{constant}$; and r is the radial distance from magnetic axis. Natural coordinates (NCC) are (ψ, θ, φ) where $\psi = B_0 r^2 / 2$ where B_0 is the magnetic field on magnetic axis, and $\theta = \tan^{-1}(y/x)$ [7]. NCC can be mapped to physical coordinates. Physical coordinates give the position in real physical space. The most efficient way to study the homoclinic tangle of separatrix is to utilize a symplectic map [5].

The simple map [8] is the simplest map that has the magnetic topology of a single-null divertor tokamak. The separatrix and the open surfaces of the simple map have an open, non-compact topology. On the other hand, the separatrix and the open surfaces of the symmetric quartic map (SQM) [9] have a closed and compact topology. The symmetric quartic map is the simplest map that has the magnetic topology of a single-null divertor tokamak with a closed and compact topology as contrasted with the simple map. Here we calculate the homoclinic tangle of the separatrix of the SQM using the map parameter k of the SQM to represent the magnetic asymmetries as is done for the Standard Map [10]. The long term goal of this work is to assess the effects of closed and open topologies of the separatrix on the homoclinic tangles in divertor tokamaks. For this purpose the coefficients of the generating function of the SQM are chosen such that the height, width, and the poloidal flux inside the separatrix are same as in the simple map. Under these conditions, the generating function for the SQM in NCC is

$$S(\psi, \theta) = a(\theta)\psi + b(\theta)\psi^{3/2} + c(\theta)\psi^2.$$

Here $a(\theta) = 1 + \frac{5}{12} \sin^2(\theta)$, $b(\theta) = -\frac{128}{81} \sin^3(\theta)$ and $c(\theta) = \frac{128}{243} \sin^4(\theta)$. The map equations are then given by

$$\psi_{n+1} = \psi_n - k \frac{\partial S(\psi_{n+1}, \theta_n)}{\partial \theta_n}, \quad \theta_{n+1} = \theta_n + k \frac{\partial S(\psi_{n+1}, \theta_n)}{\partial \psi_{n+1}} \quad \text{and} \quad \varphi_{n+1} = \varphi_n + k.$$

The map parameter k represents the magnetic asymmetries. For the backward map, the θ -equation is first solved for θ_n , and then the ψ -equation is solved for ψ_n . When the separatrix

manifold is mapped forward and backward a single toroidal circuit, the forward and backward manifolds meet in the $\varphi=0$ plane and form homoclinic tangle. The ideal separatrices of the simple map and the SQM are shown in Fig. 1. Homoclinic tangles are shown in Figs. 2-6.

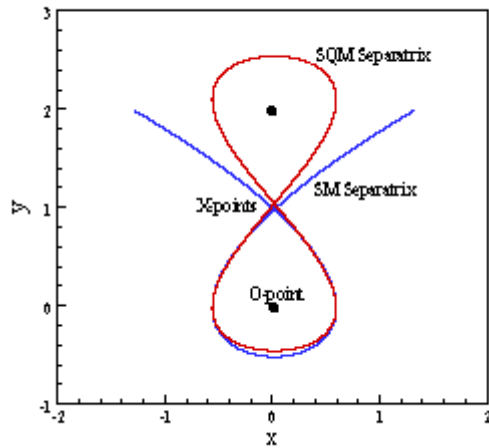


Fig. 1. Ideal separatrix for the simple map and the symmetric quartic map.

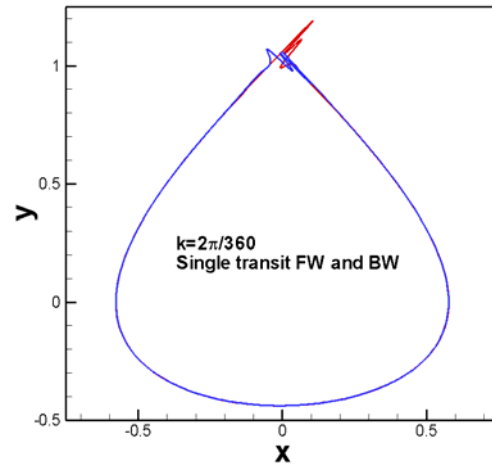


Fig. 2. Homoclinic tangle of the SQM separatrix for $k=2\pi/360$ after a single toroidal circuit in the $\varphi=0$ plane.

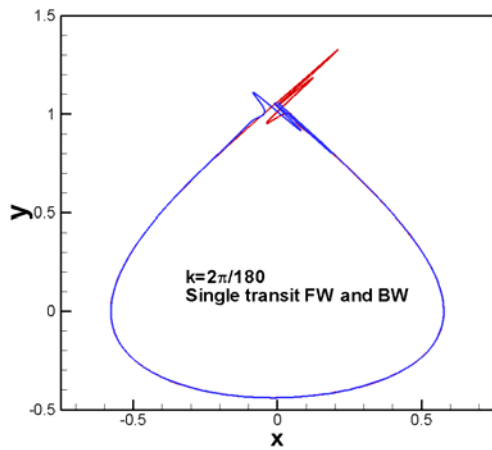


Fig. 3. Homoclinic tangle of the SQM separatrix for $k=2\pi/180$ after a single toroidal circuit in the $\varphi=0$ plane.

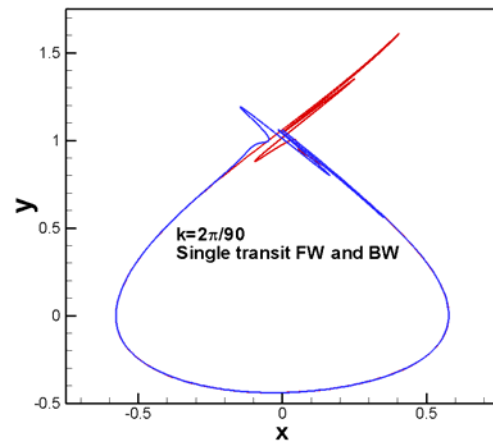


Fig. 4. Homoclinic tangle of the SQM separatrix for $k=2\pi/90$ after a single toroidal circuit in the $\varphi=0$ plane.

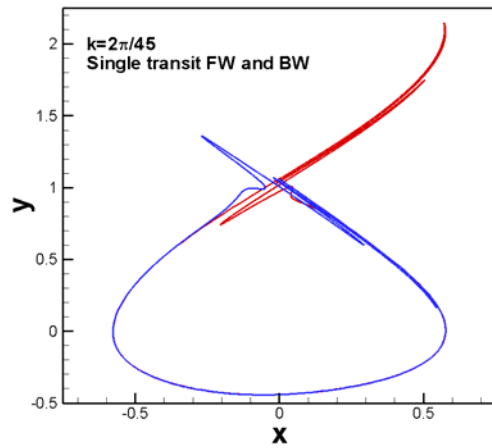


Fig. 5. Homoclinic angle of the SQM separatrix for $k=2\pi/45$ after a single toroidal circuit in the $\varphi=0$ plane.

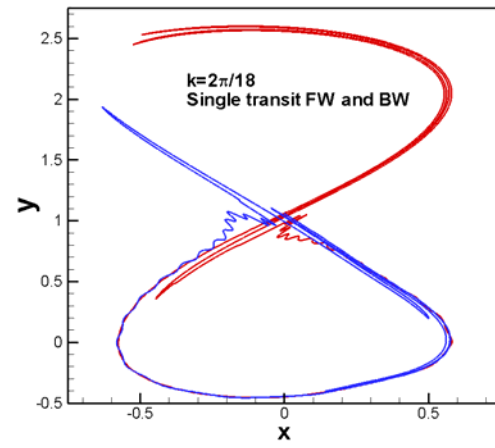


Fig. 6. Homoclinic angle of the SQM separatrix for $k=2\pi/18$ after a single toroidal circuit in the $\varphi=0$ plane.

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