Electromagnetic homogenization in a blob-populated scrap-off layer of magnetically confined plasmas(*)

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Radio frequency waves are of tantamount importance for heating and current drive in magnetized fusion plasmas. The scattering process of these waves by a multitude of density fluctuations, such as blobs in the edge region, is studied by homogenizing the part of the edge region which is populated by an ensemble of ellipsoidal blobs immersed in the ambient plasma. For that region, the effective permittivity tensor is formulated on the basis of the depolarization dyadic. The ambient density is considered as step-wise constant. In general, the interfaces between the slab characterized by the effective permittivity tensor and the ambient plasma are not necessarily aligned with the ambient magnetic field which is considered as homogeneous.

The widely used homogenization formalism has some serious shortcomings. The first one is that the depolarization dyadic that is used does not take into account the size of the blobs. The second and most important one is the main work assumption that the size of the blobs is much smaller than the wavelength of the incoming beam. This excludes a large portion of the electromagnetic radiation spectrum, and renders the formalism inaccurate. Our focus is to generalize the method excluding this assumption.

Description of the method

From this point of view, the waves propagate in three regions. The first one is homogeneous and anisotropic and consists of ions with number density n_1. Then, the incident wave propagates in an intermediate dielectric mixture plasma region which consists of a homogeneous anisotropic environment in which the ions have number density n_3 and anisotropic elliptical inclusions (blobs) with ion number density n_B. After getting through this intermediate region, the wave propagates in a homogeneous anisotropic region consisting of ions with number density n_2. Furthermore, the axis of the intermediate region is not necessarily parallel to the external imposed magnetic field $\mathbf{B}_0$ (Fig. 1).

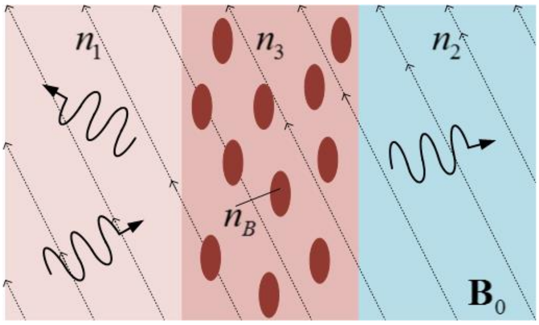

Figure 1: A simplified depiction of the problem – the three propagation regions

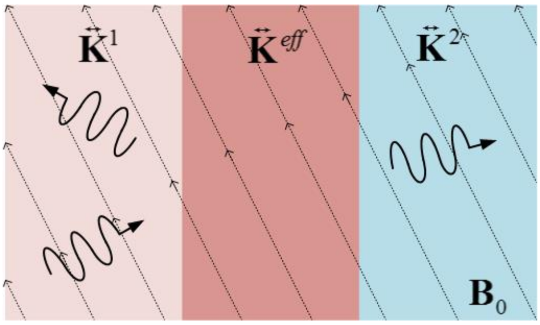

Figure 2: Intermediate region is considered as a dielectric region with calculated effective tensor $\overleftrightarrow{\mathbf{K}}^{eff}$

So, in the intermediate propagation region, there are anisotropic elliptical inclusions with an arbitrary permittivity tensor $\overleftrightarrow{\mathbf{K}}^B$ located in a homogeneous anisotropic environment with a permittivity tensor $\overleftrightarrow{\mathbf{K}}^P$. This problem, can be simplified by replacing the region of plasma with elliptical blobs with a dielectric region with effective permittivity tensor $\overleftrightarrow{\mathbf{K}}^{eff}$ (Fig. 2) [1], [2]

where $\overleftrightarrow{\mathbf{K}}^{eff,B,P} = \begin{pmatrix} K_\perp^{eff,B,P} & -jK_\times^{eff,B,P} & 0 \\ jK_\times^{eff,B,P} & K_\perp^{eff,B,P} & 0 \\ 0 & 0 & K_\parallel^{eff,B,P} \end{pmatrix}$.

It is necessary to insert some functions for further study. Let $\overleftrightarrow{\mathbf{G}}(\boldsymbol{\rho}-\boldsymbol{\rho}')$ be the dyadic Green's function of the differential operator, $\mathbf{e}_\omega$ the electric field vector (in our special case of gyrotropic materials, the magnetic field will remain unchanged) and denote by $\hat{\overleftrightarrow{\mathbf{G}}}(\mathbf{q}-\mathbf{q}')$ the spatial Fourier transform of the Green's function, with $q=kc/\omega$.

The analytical expression of $\hat{\overleftrightarrow{\mathbf{G}}}(\mathbf{q}-\mathbf{q}')$ is given in [3]. Next, we introduce the vector $\mathbf{e}_\omega^P$, which is the solution of the differential equation the fields must satisfy when the entire medium has properties "*P*" (when there are no blobs in the plasma). The differential equation that has to be satisfied is:

$$\mathbf{e}_\omega(\boldsymbol{\rho}) = \mathbf{e}_\omega^P(\boldsymbol{\rho}) + \int_{\substack{\vec{\rho}'\in V' \\ V_E \subset V'}} dV'\,\overleftrightarrow{\mathbf{G}}_P(\boldsymbol{\rho}-\boldsymbol{\rho}')\cdot\vec{\mathbf{Q}}_{equiv}(\boldsymbol{\rho}') = \mathbf{e}_\omega^P(\boldsymbol{\rho}) + j\int_{\vec{\rho}'\in V_E} dV'\,\overleftrightarrow{\mathbf{G}}_P(\boldsymbol{\rho}-\boldsymbol{\rho}')\cdot\left(\overleftrightarrow{\mathbf{K}}_P - \overleftrightarrow{\mathbf{K}}_B\right)\cdot\mathbf{e}_\omega(\boldsymbol{\rho}')$$

where V_E is the volume that an ellipsoidal inclusion occupies. This equation can be solved via Liouville-Neumann infinite series. First the equation needs to be Fourier transformed, because in general the dyadic Green's function is not known, but its Fourier transform is [3]. This gives $\hat{\mathbf{e}}_\omega = \hat{\mathbf{e}}_\omega^P + j\hat{\overleftrightarrow{\mathbf{G}}}_P \cdot \left(\overleftrightarrow{\mathbf{K}}^P - \overleftrightarrow{\mathbf{K}}^B\right) \cdot \hat{\mathbf{e}}_\omega \hat{H}_E$ where $\hat{H}_E$ is the Fourier transform of the Heaviside step function inside the ellipsoidal blob. Introducing a matrix $\overleftrightarrow{\mathbf{E}}$ which transforms an ellipsoidal region into a spherical one, the differential equation becomes

$$\hat{\mathbf{e}}_\omega^P(\mathbf{q}) = \int_{\mathbf{q}'} d^3\mathbf{q}'$$

$$\left\{ \delta(\mathbf{q}-\mathbf{q}')\overleftrightarrow{\mathbf{I}} - j\frac{\rho_0}{2\pi^2} \frac{\sin\left[\rho_0 \left|\overleftrightarrow{\mathbf{E}}^{-1}\cdot(\mathbf{q}-\mathbf{q}')\right|\right] - \cos\left[\rho_0 \left|\overleftrightarrow{\mathbf{E}}^{-1}\cdot(\mathbf{q}-\mathbf{q}')\right|\right]}{\left|\overleftrightarrow{\mathbf{E}}^{-1}\cdot(\mathbf{q}-\mathbf{q}')\right|^2} \hat{\overleftrightarrow{\mathbf{G}}}_P(\mathbf{q}) \cdot \left(\overleftrightarrow{\mathbf{K}}^P - \overleftrightarrow{\mathbf{K}}^B\right) \right\} \cdot \hat{\mathbf{e}}_\omega(\mathbf{q}')$$

Ultimately, one is led to the following differential equation for the effective permittivity tensor:

$$(1-\sigma)\frac{d\overleftrightarrow{\mathbf{K}}^{eff}}{d\sigma} \simeq \overleftrightarrow{\mathbf{K}}^B - \overleftrightarrow{\mathbf{K}}^{eff}(\sigma) + (\overleftrightarrow{\mathbf{K}}^B - \overleftrightarrow{\mathbf{K}}^{eff}(\sigma))[\varepsilon \overleftrightarrow{\mathbf{F}}_1(\mathbf{q},\mathbf{q}';\overleftrightarrow{\mathbf{K}}^{eff}(\sigma)) + \varepsilon^2 \overleftrightarrow{\mathbf{F}}_2(\mathbf{q},\mathbf{q}';\overleftrightarrow{\mathbf{K}}^{eff}(\sigma))]$$

where $\overleftrightarrow{\mathbf{F}}_1$ and $\overleftrightarrow{\mathbf{F}}_2$ are functions of the volume fraction σ which arise from the integrations in the q-space, and the scaling parameter $\varepsilon \approx \frac{R_0}{\lambda_q}\frac{|\Delta \mathrm{n}|}{n}$ is proportional to the ratio of the blob size to the incoming wavelength multiplied by the relative density contrast among the blobs and the ambient plasma. For frequencies in the range of 170 GHz (EC) the wavelength λ_q is around *2* mm , then the blob size can be as large as 6 mm for a density contrast of about 20%. For lower frequencies (such as LH) the approximation can easily accommodate larger blobs and density contrasts up to 0.6 cm at 170 GHz.

The above differential equation can be solved by decomposing the effective permittivity tensor as $\overleftrightarrow{\mathbf{K}}^{eff} = \overleftrightarrow{\mathbf{K}}^{eff}_{(0)} + \varepsilon\overleftrightarrow{\mathbf{K}}^{eff}_{(1)} + \varepsilon^2\overleftrightarrow{\mathbf{K}}^{eff}_{(2)} + ...$ and solving each equation separately. The zeroth order equation can be easily solved analytically, and gives $\overleftrightarrow{\mathbf{K}}^{eff}_{(0)} = \sigma\overleftrightarrow{\mathbf{K}}^B + (1-\sigma)\overleftrightarrow{\mathbf{K}}^P$. The complexity of the differential equation rises proportionally to the order of the approximation e.g. the first order equation involves integrals of six variables , the second order one integrals of nine variables etc. Inside the integrand, the dielectric tensor that is used is the zeroth order solution, else the equation is unable to be solved. Solving up to the first or (at most) second order suffices to approximate the behavior of the effective permittivity tensor.

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