Of electrostatic envelope modes and freak wave modeling in plasmas: revisiting a widespread fallacy

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Abstract

Recent theoretical considerations link the nonlinear Schorödinger equation (NLSE) to localized envelope forms, known as *breathers*, which have been proposed as prototypes for extreme-amplitude excitations (freak waves, rogue waves) in various areas, including modulated-amplitude wavepackets in plasmas. The analytical methodology for the derivation of the NLSE is briefly reviewed, and its physical implications are discussed. A widespread "shortcut" methodology, relating it to small-amplitude theories for solitary waves, is shown to be inherently flawed, as regards its application in fluid plasma theory (notwithstanding the potential value of the method in other areas, e.g. in hydrodynamics).

1. Introduction. A number of recent studies have been dedicated to extreme amplitude excitations (*freak waves*, or *rogue waves*, RWs) occurring in plasmas [1]. Expressed as localized envelope forms (*bright solitons*, or *breathers*), these are normally studied via multiscale techniques. Applied to a plasma-fluid model, the method leads to a nonlinear evolution equation in the form of a nonlinear Schrödinger equation (NLSE), describing the slowly varying envelope ψ of the electrostatic (ES) potential (ϕ) envelope [2, 3]. In an analogous (algebraically less tedious, but far less rigorous) manner, some recent studies have relied on the NLSE approach, but using small-amplitude (e.g., Korteweg – de Vries, KdV) type equations [4] as starting point.

We shall show that the latter method, however well established in water dynamics [5], is intrinsically flawed, and in fact leads to dubious physical predictions, when applied to ES plasma modes. In particular, the method predicts wavepackets propagating above the plasma "sound speed". Furthermore, it may also lead to erroneous criteria for the existence of envelope modes.

2. Modeling considerations. We start by reviewing some well known paradigms, based on generic nonlinear partial-differen- tial equations (PDEs): these are derived from plasma fluid models, in the small-amplitude approximation, via so-called *reductive perturbation theory* [4, 6]. Only one-dimensional (1D) models will be considered in the following.

Gardner equation. Let us shall consider, as starting point in our discussion, the *extended Korteweg-de Vries* (eKdV), also known as the *Gardner* equation:

$$\frac{\partial \phi}{\partial T} + A\phi \frac{\partial \phi}{\partial X} + A'\phi^2 \frac{\partial \phi}{\partial X} + B\frac{\partial^3 \phi}{\partial X^3} = 0\,, \tag{1}$$

Here, A, A' and B are real coefficients, usually given as functions of the plasma configuration [6]. It is understood that $B > 0$, while A and A' may be either positive and negative, a fact that affects the nature (polarity) of analytical solutions. In a plasma-fluid theoretical context, the Gardner equation is derived via reductive perturbation theory [6], assuming small-amplitude excitations off-equilibrium (i.e., $\phi \simeq \varepsilon\phi_1 + ...$ for the electrostatic potential, and analogous expansions for the various fluid state variables) based on a particular variable "stretching", so that the space and time variables are $X = \varepsilon(x - U_0 t)$ and $T = \varepsilon^3 t$, respectively (where $\varepsilon \ll 1$ is a small real parameter). Note, for the sake of "book-keeping", that the derivation of Eq. (1) relies on a restricting assumption, namely that the quartic nonlinearity coefficient A is small but finite, hence higher orders of nonlinearity must be resorted to: this is often referred to as a *near-critical* plasma configuration [6]. Importantly, U_0 in the above variable *Ansatz* is a real parameter which is determined during the multiscale perturbation procedure, and is *a priori* not known beforehand. However, it is found through a compatibility constraint to be equal to the plasma sound speed c_s ($= (k_B T_e/m_i)^{1/2}$, e.g., for e-i plasma, actually recovered as the small-k limit of the phase speed ω/k. Various solutions of Eq. (1) exist, in fact – most interestingly – characterized by either positive or negative pulse polarity; see e.g. in [7] and references therein.

Korteweg de-Vries (KdV) equation. The KdV equation is formally recovered from (1), upon setting $A' = 0$. Its derivation in fact relies on a different scaling, namely $X = \varepsilon^{\frac{1}{2}}(x - U_0 t)$ and $T = \varepsilon^{\frac{3}{2}} t$, to be precise; $U_0 = c_s$, as above. The well known pulse-soliton solution of the KdV eq. is of the form $\phi \sim \phi_0 \,\mathrm{sech}^2(X - VT)/L$, where the pulse's maximum amplitude ϕ_0 and the spatial extension (width) L are known functions of A and B, satisfying $\phi_0 L^2 =$constant [4].

Modified Korteweg de-Vries (mKdV) equation. The mKdV equation is recovered from Eq. (1) for $A = 0$. The exact derivation of the mKdV equation from a plasma fluid model is meaningful only at (strictly) *critical* plasma configuration (viz., $A = 0$ is satisfied by the plasma constituents) [6]. Various pulse-shaped solutions of the mKdV eq. exist.

A common feature of the above (KdV family of) equations is the fact that their solitonic solution(s) is (are) expressed in terms of the argument, say, $\xi = X - VT$; for the KdV. eq., $\xi = \varepsilon^{1/2}[x - (c_s + \varepsilon V)t]$, recovering the original (physical) space and time coordinates x and t, in the laboratory frame. The variable V is thus to be interpreted as the velocity increment (above the sound speed) of the pulse soliton. Similar consideration hold for all of the above equations.

One must therefore bear in mind that ***the KdV theory models (only) small-amplitude, weakly super-sonic soliton structures***, as ϕ ***($\sim \varepsilon$) modeled by Eq. (1) (or any of its variants) actually travels little faster than the sound speed***. *(Statement 1)*

3. Amplitude modulation formalism – scenario # 1: a rigorous approach. The Nonlinear Schrödinger equation (NLSE) can be obtained from fluid plasma models in a similar (yet not identical) method to the one described above, via a multiple scales technique. The method relies in adopting multiple space and time scales, in order to distinguish the fast (carrier wave related) scales ($X_0 = x$, $T_0 = t$) from the slower (envelope related) scales ($X_1 = \varepsilon x$, $X_0 = \varepsilon^2 x$, ...; $T_1 = \varepsilon t$, $T_0 = \varepsilon^2 t$, ...). Small deviations are considered of all plasma state variables from the equilibrium state, i.e. $\phi = 0 + \varepsilon\phi^{(1)} + \varepsilon^2\phi^{(0)} + ... = \Sigma_{n=1}^{\infty}\varepsilon^n\phi_n$, and so on. Harmonic generation is accounted for by assuming, at each order n, $\phi_n = \Sigma_{l=-\infty}^{\infty}\phi_l^{(n)}(X_{1^+}, T_{1^+})\, e^{il(kx-\omega t)}$.

The (carrier wave) dispersion relation (DR) $\omega = \omega(k)$ is obtained, relating the fundamental frequency ω to the wavenumber k. A tedious analytical procedure [3] leads to the equation:

$$i\frac{\partial\psi}{\partial\tau} + P\frac{\partial^2\psi}{\partial\zeta^2} + Q \mid \psi \mid^2 \psi = 0\,, \tag{2}$$

where $\psi = \phi_1^{(1)}$ and the (slow) independent variables are $\zeta = X_1 - v_g T_1 = \varepsilon(x - v_g t)$ and $\tau = T_2 = \varepsilon^2 t$. In Eq. (2), the dispersion coefficient is $P = \omega''(k)/2$, while the nonlinearity coefficient Q is obtained analytically, and may be a tedious function of the wavenumber k, also involving intrinsic plasma parameters (e.g. density or temperature of various plasma constituents).

The NLS equation (2) is an integrable equation [8] that possesses exact solutions, in the form of envelope solitons of various types [3]. A class of solutions (obtained if $PQ > 0$), known as *breathers*, have been proposed as prototypical analytical models for RWs [9]. We won't go into detail here; it suffices to say that these solutions are given by analytical functions of a travelling coordinate $\xi = \zeta - \tilde{u}\tau$; therefore, the free variable $\tilde{u}$ denotes the velocity increment *above* the group velocity v_g. (Recall that the variables entering the NLSE are in fact ζ and τ, i.e. *not* x, t.)

Importantly, the bright soliton solution of Eq. (2) [3] (and, in fact, the entire family of *breather* solutions) are functions of a travelling coordinate, say, $\zeta - \tilde{u}\tau$, which may be interpreted as $\varepsilon[x - (v_g + \varepsilon\tilde{u})t]$ in terms of the original coordinates. ***The (envelope) amplitude ψ – as described by (2) – therefore travels at (or, in fact, slightly above) the group velocity v_g ($\neq c_s$).(Statement 2)***

4. Amplitude modulation formalism – scenario # 2: a flawed approach. Various theoretical studies in plasmas have undertaken a derivation of the NLSE (2) based on the small-amplitude Eq. 1 (or variants thereof) [10]. In more recent cases, e.g. [11, 12], the authors have proceeded by proposing a freak-wave related interpretation of the erroneous NLS equation derived. A very interesting experiment on ES freak waves in plasmas with negative ions [13] has actually based its interpretation on the assumption that freak wave formation relies on the reali-

sation of a critical plasma configuration, so that the associated mKdV equation [10] may be the (only) relevant starting point for the prediction of the occurrence of freak waves. We argue that *this methodology is misleading and clearly gives wrong results*. In order to see the intrinsic flaw in this approach, we may consider a simple case study. Let us consider the derivation of the NLS equation from the Gardner eq. (1), by making use of the formalism in Sec. 3 above. The dispersion relation derived upon linearizing (1) by setting $\phi \sim e^{i(k_1 X - \omega_1 T)}$ is $\omega_1 = -Bk_1^3$, leading to $v_g = -3Bk_1^2$ and $P = \omega_1''(k_1)/2 = -3k_1$. As a first remark, both v_g and P are negative $\forall k_1$, *regardless of* the particular plasma specification considered in deriving (1) (a rather dubious fact). P, is here always negative. Further analysis leads to an expression for $Q = k_1 \left(\frac{A^2}{6Bk_1^2} - A' \right)$.

1st paradox: let us assume that $A' = 0$ (*KdV* equation) for a while. One finds that $P < 0 < Q \Rightarrow PQ < 0$ for all values of A and B (regardless of the plasma configuration, that is!), hence, there is no modulational instability whatsoever. This is a wrong result, as the NLSE is known to give both stable and unstable regions, even for the simplest (e.g. electron-ion) plasmas [2].

2nd paradox: let us now assume that $A = 0$ (*mKdV* equation). One easily draws the conclusion that $PQ > 0$, hence modulational instability (and breather, or freak wave, occurrence) is always possible [3], $\forall k_1$. This is clearly not a generally valid result. This method, adopted in [10], was later apparently lent to the interpretation of the (unique of its kind) "critical-density" negative-ion plasma experiment on freak waves by Bailung et al [13].

Upon careful inspection, one realizes that the "KdV waveform" $e^{i(k_1 X - \omega_1 T)}$ is actually tantamount to $e^{i\varepsilon^{1/2}[k_1 x - (c_s k_1 - \varepsilon B k_1^3)t]}$ (cf. the KdV variable stretching in Sec. 2 and Statements 1-2 above). The KdV equation has been formulated in a reference frame moving at the sound speed, by balancing nonlinearity and dispersion, thus any derivation of the NLSE is of limited value.

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