Stability criteria of ideal magnetohydrodynamic plasmas with flows

Tommaso Andreussi [1], Philip J. Morrison [2], Francesco Pegoraro [3]

[1] *Space Propulsion Division, Sitael S.p.A., Pisa, Italy*

[2] *Inst. for Fusion Studies Phys. Dept., The University of Texas at Austin, Austin, USA*

[3] *Phys. Dept., University of Pisa, Pisa, Italy*

Abstract

The study of the stability of MHD plasma equilibria with stationary flows requires a generalization of the standard $\delta\mathscr{W}$ approach that is used for static configurations. This extension is best performed [1] by looking at the functional $\delta\mathscr{W}$ not as a quadratic form derived from the linearized MHD equations but as the second order variation of the Hamiltonian functional $\mathscr{H}$ [2] that describes the full dynamics of a dissipationless MHD plasma. With this approach the Hermitian property follows automatically. The second variation of the Hamiltonian determines the MHD plasma stability and can be computed either in Lagrangian or in Eulerian variables. If stationary equilibrium flows are present the two procedures follow somewhat different paths. Here we illustrate these differences and exemplify them in the case of a rotating pinch [3].

The Hamiltonian functional $\mathscr{H}$, equilibria with flows and time dependent relabelling

In Eulerian variables $\mathscr{H} = \int d\mathbf{x}\,[\rho|\mathbf{v}|^2/2 + \rho\,U(s,\rho) + |\mathbf{B}|^2/8\pi]$, with $\rho(x,t)$ the density, $\mathbf{v}(x,t)$ the fluid velocity, $U = U(s,\rho)$ and $s(x,t)$ the internal energy and entropy per unit mass and $\mathbf{B}(x,t)$ the magnetic field. The variables $Z = \rho, \mathbf{v}, s, \mathbf{B}$ are noncanonical and their equations of motion, $\partial Z/\partial t = \{Z, H\}_Z$ involve noncanonical Poisson brackets [4] whose general form is $\{F,G\} = \int dx(\delta F/\delta Z)\cdot\mathbb{J}\cdot(\delta G/\delta Z)$ with $\mathbb{J}$ an anti-selfadjoint operator. The degeneracy of the noncanonical brackets gives rise to Casimir invariant functionals C_i that satisfy $\{C_i, F\} = 0$ for all functionals F. The Casimir invariance implies that the system evolution is restricted to subdomains (foliations) of the space of the Eulerian variables Z.

The map [5] from the Lagrangian variables $\mathbf{q}(\mathbf{a},t), \pi(\mathbf{a},t)$ to the Eulerian variables Z: $\rho(\mathbf{x},t) = \rho_0(\mathbf{a})/J(\mathbf{a},t), s(\mathbf{x},t) = s_0(\mathbf{a}),\ v_i(\mathbf{x},t) = \pi_i(\mathbf{a},t)/\rho_0(\mathbf{a}),\ B^i(\mathbf{x},t) = [\partial q^i(\mathbf{a},t)/\partial a_j]\,[B_{0j}(\mathbf{a})\{J(\mathbf{a},t)\}]$, all evaluated at $\mathbf{a} = \mathbf{q}^{-1}(\mathbf{x},t)$ with $J = |\partial q^i/\partial a^j|$, gives $\mathscr{H}$ in Lagrangian variables $\mathscr{H}[\mathbf{q},\pi] = \int d\mathbf{a}\,[\pi_i\pi^i/2\rho_0 + \rho_0 U(s_0, \rho_0/J) + (\partial q_i/\partial a^k)\,(\partial q^i/\partial a^\ell)\,(B_0^k B_0^\ell/8\pi J)]$, together with the canonical equations of motion $\dot{\pi}_i = \{\pi_i, H\} = -\delta H/\delta q^i$ and $\dot{q}^i = \{q^i, H\} = \delta H/\delta \pi_i$.

Eulerian equilibria are extrema of the Casimir constrained Hamiltonian $\mathfrak{F} = \mathscr{H} + \Sigma_i C_i$. Different choices of this “energy Casimir” functional lead to different equilibria. Explicit expressions for the Casimirs sufficient to describe general families of equilibria with flows may be difficult to obtain, thus this method is generally applied to geometrically symmetric equilibria.

Restricting to Dynamically accessible (DA) variations that are generated by the noncanonical Poisson brackets [6] bypasses this difficulty while ensuring that kinematical constraints are satisfied. The first order DA variations are: $\delta\rho_{\rm da} = \nabla\cdot(\rho\mathbf{g}_1)$, $\delta\mathbf{v}_{\rm da} = \nabla g_3 + s\nabla g_2 + (\nabla\times\mathbf{v})\times\mathbf{g}_1 + \mathbf{B}\times(\nabla\times\mathbf{g}_4)/\rho$, $\delta s_{\rm da} = \mathbf{g}_1\cdot\nabla s$, $\delta\mathbf{B}_{\rm da} = \nabla\times(\mathbf{B}\times\mathbf{g}_1)$ with $\mathbf{g}_1$, g_2, g_3, and $\mathbf{g}_4$ arbitrary.

Since Eulerian equilibria with flows are not Lagrangian equilibria we introduce a time dependent relabelling [1] $\mathbf{a} = \mathfrak{A}(\mathbf{b},t)$, with inverse $\mathbf{b} = \mathfrak{B}(\mathbf{a},t)$ and the new dynamical variables and Hamiltonian $\Pi(\mathbf{b},t) = \mathfrak{J}\,\pi(\mathbf{a},t)$, $\mathbf{Q}(\mathbf{b},t) = \mathbf{q}(\mathbf{a},t)$ $\tilde{\mathscr{H}}[\mathbf{Q},\Pi] = \mathscr{H} - \int d\mathbf{b}\,\Pi\cdot(\mathbf{V}\cdot\nabla_b\mathbf{Q})$, with $\mathbf{V}(\mathbf{b},t) = \dot{\mathfrak{B}}\circ\mathfrak{B}^{-1} = \dot{\mathfrak{B}}(\mathfrak{A}(\mathbf{b},t))$ the label velocity, $\nabla_b = \partial/\partial\mathbf{b}$, and $\mathfrak{J} = \det(\partial a^i/\partial b^j)$.
Setting $\mathbf{V}(\mathbf{b},t) = \mathbf{v}_e(\mathbf{b})$, where $\mathbf{v}_e(\mathbf{b})$ corresponds to an Eulerian equilibrium state, relabelling allows us to express in Lagrangian variables stationary equilibria, which in these variables would be time dependent, as time-independent when referred to moving labels.

Stability: energy Casimir, Dynamical Accessible and Lagrangian

For energy Casimir equilibria a sufficient stability condition follows if $\delta^2\mathfrak{F}$ is positive definite. For perturbations invariant along z we have $\delta^2\mathfrak{F}[Z_e;\delta Z_s] = \int d\mathbf{x}[a_1|\delta\mathbf{S}|^2 + a_2(\delta Q)^2 + a_3(\delta R_z)^2 + a_4|\delta\mathbf{R}_\perp|^2 + a_5(\delta\psi)^2]$ where ψ is the magnetic flux function and $\delta\mathbf{S}, \delta\mathbf{R}, \delta Q, \delta\psi$ are linear combinations of $\delta\mathbf{v}, \delta\mathbf{B}, \delta\rho, \delta s$. Extremizing over all variables except $\delta\psi$ and back substituting gives $\delta^2\mathfrak{F}[Z_e;\delta\psi] = \int d\mathbf{x}[b_1|\nabla\delta\psi|^2 + b_2(\delta\psi)^2 + b_3|\mathbf{e}_\psi\times\nabla\delta\psi|^2]$, $\mathbf{e}_\psi = \nabla\psi/|\nabla\psi|$. Here the coefficients a_i and b_i depend on space through the equilibrium.

For a Lagrangian equilibrium in moving labels we expand $\mathbf{Q} = \mathbf{Q}_e(\mathbf{b},t) + \eta(\mathbf{b},t)$, $\Pi = \Pi_e(\mathbf{b},t) + \pi_\eta(\mathbf{b},t)$, with η and π_η relabelled canonical pairs and obtain $\delta^2\mathscr{H}_{\rm la}[Z_e;\eta,\pi_\eta] = \int d\mathbf{x}[|\pi_\eta - \rho_e\mathbf{v}_e\cdot\nabla\eta|^2/\rho_e + \eta\cdot\mathfrak{V}_e\cdot\eta]/2$, where the operator $\mathfrak{V}_e$ has no explicit time dependence and $\delta^2\mathscr{W}_{\rm la}[Z_e;\eta] = \int d\mathbf{x}\,\eta\cdot\mathfrak{V}_e\cdot\eta/2 = \int d\mathbf{x}[\rho_e(\mathbf{v}_e\cdot\nabla\mathbf{v}_e)\cdot(\eta\cdot\nabla\eta) - \rho_e|\mathbf{v}_e\cdot\nabla\eta|^2]/2 + \delta^2\mathscr{W}[Z_e\eta]$ is identical to the functional obtained by Frieman and Rotenberg [7]. Due to the arbitrariness of π_η, which does not contribute to $\delta^2 W_{\rm la}$, the quadratic term $|\pi_\eta - \rho_e\mathbf{v}_e\cdot\nabla\eta|^2$ can be put equal to zero and a sufficient condition for stability is $\delta^2\mathscr{W}_{\rm la}[Z_e;\eta] > 0$ for any η.

Dynamically accessible stability is assessed by expanding the Hamiltonian in Eulerian variables to second order using the dynamically accessible constraints to this order: $\delta^2 H_{\rm da}[Z_e;\mathbf{g}] = \int d\mathbf{x}\,\rho|\delta\mathbf{v}_{\rm da} - \mathbf{g}_1\cdot\nabla\mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{g}_1|^2 + \delta^2 W_{\rm la}[\mathbf{g}_1]$. If $\delta\mathbf{v}_{\rm da}$ were independent and arbitrary we could use it to nullify the first term. Then setting $\mathbf{g}_1 = -\eta$, we would see that dynamically accessible stability is identical to Lagrangian stability. However in general there is not sufficient freedom in the generating functions to cancel the positive definite first term (see also [8]).

Comparison between the three different stability criteria

Because different constraints are imposed, stability conditions take different forms when derived within the Lagrangian, Eulerian (energy-Casimir), or dynamical accessible frameworks. Different perturbations are associated with the three expressions and can be written as

$$\begin{cases} \delta\rho_{\rm la} &= -\nabla\cdot(\rho\eta) \\ \delta\mathbf{v}_{\rm la} &= \frac{\partial\eta}{\partial t} + \mathbf{v}\cdot\nabla\eta - \eta\cdot\nabla\mathbf{v} \\ \delta s_{\rm la} &= -\eta\cdot\nabla s \\ \delta\mathbf{B}_{\rm la} &= -\nabla\times(\mathbf{B}\times\eta) \end{cases} \qquad \begin{cases} \delta\rho_{\rm ec} \\ \delta\mathbf{v}_{ec} \\ \delta s_{\rm ec} \\ \delta\mathbf{B}_{ec} \end{cases} \qquad \begin{cases} \delta\rho_{\rm da} &= -\nabla\cdot(\rho\mathbf{g}_1) \\ \delta\mathbf{v}_{\rm da} &= \mathbf{X} + \mathbf{v}\cdot\nabla\mathbf{g}_1 - \mathbf{g}_1\cdot\nabla\mathbf{v} \\ \delta s_{\rm da} &= -\mathbf{g}_1\cdot\nabla s \\ \delta\mathbf{B}_{\rm da} &= -\nabla\times(\mathbf{B}\times\mathbf{g}_1) \end{cases}$$

where $\mathbf{X} = 2(\mathbf{v}\cdot\nabla)\mathbf{g}_1 + \mathbf{v}\times(\nabla\times\mathbf{g}_1) + s\nabla g_2 + \nabla g_3 + \frac{1}{\rho}\mathbf{B}\times(\nabla\times\mathbf{g}_4)$.
The Lagrangian perturbations $\mathfrak{P}_{\rm la}$ are constrained, while for the energy-Casimir expression the perturbations $\mathfrak{P}_{\rm ec}$ are entirely unconstrained (provided they satisfy the translation symmetry). The dynamically accessible perturbations are constrained. The following inclusions apply:

$$\mathfrak{P}_{\rm da} \subset \mathfrak{P}_{\rm la} \subset \mathfrak{P}_{\rm ec}\,, \qquad \text{which implies} \qquad \mathfrak{stab}_{ec} \Rightarrow \mathfrak{stab}_{\rm la} \Rightarrow \mathfrak{stab}_{\rm da}\,.$$

Dynamically accessible stability is the most limited because its perturbations are the most constrained, while energy-Casimir stability is the most general, when it exists, because its perturbations are not constrained at all.

Explicit comparison for a rigid rotating isothermal configuration

Consider a rotating plasma equilibrium where all quantities depend only on r : $\mathbf{B} = B_z(r)\hat{\mathbf{z}} + B_\phi(r)\hat{\phi}$, with $B_\phi = \hat{\phi}\cdot\nabla\psi\times\hat{\mathbf{z}}$, $\mathbf{v} = v_\phi(r)\hat{\phi}$, $\rho = \rho(r), s = s(r)$. The generalized Grad-Shafranov (GGS) equation for $\psi(r)$ involves the poloidal Alfvèn Mach number $\mathscr{M}$ and reads

$$\frac{1}{r}\frac{d}{dr}\Big(\frac{1-\mathscr{M}^2}{4\pi} rB_\phi\Big) - \frac{1}{\psi_r}\frac{d}{dr}\Big(p + \frac{B_z^2}{8\pi}\Big) + \frac{d}{dr}\Big(\frac{\mathscr{M}^2}{4\pi}B_\phi\Big) = 0, \quad \mathscr{M}(r) = [4\pi\rho(r)v_\phi^2(r)/B_\phi^2(r)]^{1/2}.$$

We set in dimensionless units $B_z(r) = B_z$, $B_\phi(r) = B_0 r$, and $v_\phi(r) = \Omega r$ with B_z, B_0, Ω constants. Since the plasma is isothermal $p(r)$ and $\rho(r)$ are linearly related. Solving GGS for $p(r)$ yields a one-parameter family of equilibria $\hat{p}(r) = (2/w^2)\,[1-(1-w^2/2)\exp(w^2r^2/2)]$, with $w = \Omega r_0/c_s$ $(w^2/2 < 1)$, c_s the sound velocity, $\hat{p}(0) = 1$, $\hat{p}(\bar{r}) = 0$ for $\bar{r}^2 = -(2/w^2)\ln(1-w^2/2)$. A uniform B_z field does not alter these equilibrium configurations but affects their stability.

Comparison results

In Ref.[3] we performed an analytical comparison of the stability boundaries in the w, $\hat{b} = B_z/B_0$ plane for translationally invariant perturbations illustrating the different steps in the procedure including the derivation of the equilibrium from the first variation of the Hamiltonian in

the three different formulations and the implementation of the time-dependent relabelling.
For the chosen rotating equilibrium, the Lagrangian and the dynamically accessible approaches lead to equivalent conditions. The constraints obeyed by the dynamically accessible perturbations in the presence of flows lead to a stabilizing term that does not vanish for azimuthally symmetric perturbations but that does not modify the stability analysis since azimuthally symmetric perturbations are found to be stable even within the Lagrangian framework.
The minimization of $\delta^2 W_{\text{la}}$ leads to the study of the positivity of a 3×3 matrix (a 4×4 matrix for $B_z \neq 0$ as η_z is no longer decoupled) function of the equilibrium quantities for $|m| = 1$ perturbations. A necessary and sufficient condition for the positivity of this matrix is provided by the Sylvester criterion which yields $w^2 < 1/2$ for $B_z = 0$ and $w^2 B_z^2 < 1$ for $B_z \neq 0$ and $w^2 \to 0$ and $B_z^2/B_0 < 1/3$, for $w^2 \to 1/2^-$. A partial minimization procedure with respect to η_ϕ (to η_z and η_ϕ for $B_z \neq 0$) leads to less restrictive conditions: $w^2 \lesssim 0.62$ for $B_z = 0$ and $w^2 \lesssim 0.46$ choosing, e.g., $B_z/B_0 = 1$. Even lesser restrictive conditions could be found by solving the Euler-Lagrange equation for η_r obtained via variation of the resulting "reduced" $\delta^2 \tilde{W}_{\text{la}}$ subject to the constraint of $\int r dr |r\eta_r|^2$.

Extremization of the energy-Casimir functional over all variables except $\delta\psi$ leads to sufficient stability bounds on w^2 that, as in the Lagrangian case, become stricter as B_z^2 increases. As predicted, these bounds are in general more restrictive than those found within the Lagrangian framework, as shown, e.g., by considering again $B_z^2 = 1$, in which case we find $w^2 \lesssim 0.31$.

Conclusions and remarks

The methods and the three different approaches to the study of the stability of a magnetized plasma equilibrium with steady flows described in Ref. [1], tested on an example in Ref. [3] and recalled above, are of general utility: they apply to all important dissipationless plasma models, kinetic as well as fluid, and can be extended to extended magnetofluid models [9].

References

[1] T. Andreussi, P.J. Morrison, and F. Pegoraro, Phys. Plasmas, **20**, 092104 (2013); *ibid* Phys. Plasmas, **22**, 039903 (2015).
[2] P.J. Morrison and J. M. Greene, Phys. Rev. Lett., **45**, 790 (1980).
[3] T. Andreussi, P.J. Morrison, and F. Pegoraro, Phys. Plasmas **23**, 102112 (2016).
[4] T. Andreussi, P.J. Morrison, and F. Pegoraro, Plasma Phys. Controlled Fusion 52, 055001 (2010).
[5] P.J. Morrison, Rev. Mod. Phys., **70**, 467 (1998).
[6] P.J. Morrison and D. Pfirsch, Phys. Rev., A 40, 3898 (1989).
[7] E. Frieman and M. Rotenberg, Rev. Mod. Phys., **32**, 898 (1960).
[8] E. Hameiri, Phys. Plasmas, **10**, 2643 (2003).
[9] D.A. Kaltsas, G. N. Throumoulopoulos, and P. J. Morrison, arXiv:1705.10971v1 (2017).