Nonlinear wave-particle dynamics in chorus excitation

Fulvio Zonca[1,2], Xin Tao[3,4] and Liu Chen[2,5]

[1] *ENEA, C. R. Frascati, Via E. Fermi 45, 00044 Frascati (Roma), Italy*

[2] *IFTS and Department of Physics, Zhejiang University, Hangzhou, 310027 P.R. China*

[3] *Department of Geophysics and Planetary Sciences, USTC, Hefei, P.R. China*

[4] *Collaborative Innovation Center of Astronautical Science and Technology, P.R. China*

[5] *Dept. of Physics and Astronomy, University of California, Irvine CA 92697-4575, U.S.A.*

Introduction

Nonlinear wave particle interaction during chorus wave generation has been recently shown to be a non-adiabatic process; that is, the wave-particle trapping time in the resonant phase-space structures, τ_{tr}, is typically of the same order as the characteristic nonlinear time scale τ_{NL} [1, 2]. These results shed new light on the physical processes underlying wave-particle resonance and nonlinear mode evolution with respect to previous analyses assuming $\tau_{NL} \gg \tau_{tr}$. In this work, we present an analytical study of nonlinear evolution of phase-phase space structures in support of our earlier numerical simulation results [1, 2].

We adopt a non-perturbative description [3] of the phase-space structures due to the interaction of supra-thermal electrons with the fluctuating fields produced by a quasi-periodic chorus wave. This allows us to derive the renormalized expression of supra-thermal electron distribution function in the form of a Dyson-like equation [3], which illuminates the self-consistent nonlinear evolution of resonance structures in the phase-space. In particular, we demonstrate that frequency sweeping of chorus fluctuations occurs as consequence of maximization of wave-particle power transfer; and discuss the consequence of this on the spatiotemporal features of the fluctuation spectrum.

Propagation of the chorus wave packet

The problem of parallel propagating chorus wave packet interacting with hot electrons can be approximately cast as

$$D_w |\delta \boldsymbol{E}_\perp|^2 = -\frac{4\pi i}{\omega} \delta \boldsymbol{J}_h \cdot \delta \boldsymbol{E}_\perp^* , \tag{1}$$

where we introduced the whistler wave dielectric constant, ε_w, and dispersion function, D_w,

$$\varepsilon_w = 1 + \frac{\omega_p^2}{\omega(\Omega - \omega)} , \quad D_w = \varepsilon_w - \frac{k^2 c^2}{\omega^2} . \tag{2}$$

In Eqs. (1) and (2), we treated thermal electrons as cold fluid, while supra-thermal ("hot") electrons are accounted for via the fluctuating current $\delta \boldsymbol{J}_h$ produced in response to whistler wave electric field $\delta\boldsymbol{E}_\perp$. Furthermore, $\omega_p^2 = 4\pi ne^2/m$ is the electron plasma frequency, $\Omega = eB/(mc)$ is the electron cyclotron frequency, e the positive electron charge and m the electron mass. Consistent with our earlier numerical investigations [1, 2] using the hybrid particle-in-cell code DAWN [4], we assume anisotropic Maxwellian supra-thermal electron distribution

$$f_0 = \frac{n_e}{(2\pi)^{3/2} w_{\parallel e} w_{\perp e}^2} e^{-\mathscr{E}/w_{\parallel e}^2 + A\mu B_e/w_{\perp e}^2} , \quad (3)$$

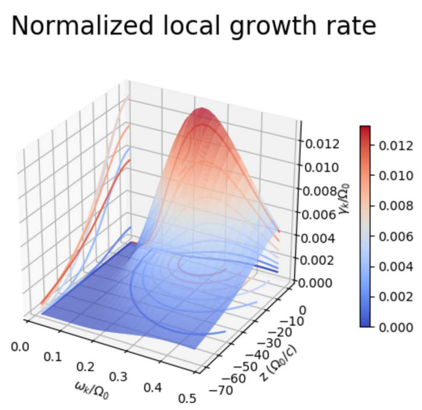

Figure 1: *Normalized linear growth rate due to hot electrons as a function of ω_k/Ω_0 and $z(\Omega_0/c)$.*

where $A \equiv w_{\perp e}^2/w_{\parallel e}^2 - 1$ is the anisotropy index, n_e is hot electron density at the equator, $w_{\parallel e}$ and $w_{\perp e}$ are, respectively, the corresponding parallel and perpendicular thermal speeds (with respect to the ambient Earth magnetic field B_e), $\mathscr{E} = v^2/2$ is the energy per unit mass and $\mu = v_\perp^2/(2B)$ is the magnetic moment. For simplicity, we also adopt a non-relativistic formulation. Non-uniformity is controlled by the dependence of B on the coordinate along the magnetic field line. Here, we take a model $B = B_e(1+\xi z^2)$, with $\xi^{-1/2}$ the non-uniformity scale length.

The elements of the whistler wave packet, which can be written as

$$\delta\boldsymbol{E}_\perp = \delta\boldsymbol{E}_{\perp 0}(z,t)\exp\left(i\int^z k(z')dz' - i\omega_k t\right) , \quad (4)$$

satisfy the WKB dispersion relation $D_w(z,k(z),\omega_k) = 0$. Meanwhile, letting $\delta\boldsymbol{E}_{\perp 0}(z,t) = |\delta\boldsymbol{E}_{\perp 0k}(z,t)|\exp i\varphi_k(z,t)$, and introducing $I_k(z,t) \equiv (\partial D_w/\partial k)|\delta\boldsymbol{E}_{\perp 0k}(z,t)|^2$, the evolution equation for $I_k(z,t)$ is

$$\left(\frac{\partial}{\partial t} + v_g\frac{\partial}{\partial z}\right) I_k(z,t) = 2\gamma_k I_k(z,t) , \quad \gamma_k = -\frac{1}{\partial D_w/\partial\omega_k}\mathbb{I}\mathrm{m}\left(\frac{4\pi i}{\omega_k}\frac{\delta\boldsymbol{J}_h\cdot\delta\boldsymbol{E}_\perp^*}{|\delta\boldsymbol{E}_{\perp 0k}(z,t)|^2}\right) , \quad (5)$$

noting $\partial_z\partial_k D_w = 0$, with $v_g = -(\partial D_w/\partial\omega_k)^{-1}\partial D_w/\partial k$ the wave packet group velocity. The evolution equation for $\varphi_k(z,t)$ is not needed here. In the linear limit, the normalized local growth rate γ_k/Ω_0 is reported in Fig. 1. Here, $\Omega_0 = eB_e/(mc)$, and we used typical parameters $\omega_p/\Omega_0 = 5$, $n_e/n = 6\times 10^{-3}$, $w_{\parallel e} = 0.2c$, $w_{\perp e} = 0.53c$, and $\xi = 8.62\times 10^{-5}\Omega_0^2/c^2$.

Dyson equation approach

The width of the linear unstable wave spectrum is determined by whistler wave dispersion relation and the anisotropic supra-thermal electron distribution function. For fixed anisotropy, i.e. $\gamma_{k0} = \gamma_k(\omega_k, z=0)$, system non-uniformity (controlled by the scale $\xi^{-1/2}$) identifies two opposite limiting behaviors: (i) weak non-uniformity, where the mode saturates before any significant non-linear frequency shift (non-uniformity induced) takes place, similar to uniform plasma system; (ii) strong non-uniformity, where particles undergo velocity change depending on the initial conditions (wave particle phase) and diffusive relaxation of particles is expected [5]. For intermediate values of non-uniformity, non-linear wave-particle interaction may be strengthened and non-perturbative non-linear interaction takes place. This transition can be understood quantitatively since, due to non-uniformity, the power exchange between chorus wave packets and supra-thermal particles is characterized by a profile $\sim (1+A\xi z^2)^{-2}$ for strong anisotropy $A \gg 1$ (cf. Fig. 1). The convective amplification of a chorus wave packet while moving through the source region, $A\xi z^2 \lesssim 1$, can, thus, be estimated by the following Padé approximation

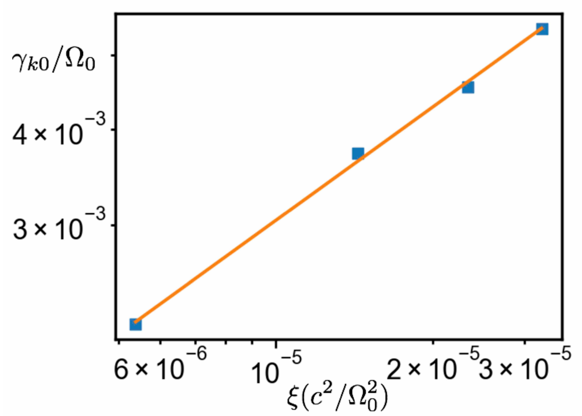

Figure 2: *Linear growth rate needed at $z=0$ for spontaneous chorus emission vs. non-uniformity control parameter $\xi(c^2/\Omega_0^2)$.*

$$\ln I_k \simeq \frac{(\pi/4)(\xi A)^{1/2} v_{g0}\tau_{NL}}{(\pi^2/16+\xi A v_{g0}^2\tau_{NL}^2)^{1/2}} \frac{2\gamma_{k0}}{(\xi A)^{1/2} v_{g0}} \,, \tag{6}$$

with v_{g0} the group velocity at $z=0$. Therefore, weak non-uniformity is identified by $\xi A v_{g0}^2\tau_{NL}^2 \ll 1$, while $\xi A v_{g0}^2\tau_{NL}^2 \gg 1$ corresponds to strong non-uniformity. Meanwhile, optimal conditions for chorus emission are obtained for $(16/\pi^2)A\xi v_{g0}^2\tau_{NL}^2 \lesssim 1$; i.e., assuming $\tau_{NL}^{-1} \sim \omega_{tr} \sim 3\gamma_{k0}$ (with ω_{tr} the wave-particle trapping frequency)

$$(\gamma_{k0}/\Omega_0) \gtrsim (12/\pi)(A\xi c^2/\Omega_0^2)^{1/2}|v_{g0}|/c \,. \tag{7}$$

We have verified the existence of this threshold condition by hybrid numerical simulations [4] of spontaneous chorus emission, reported in Fig. 2.

When we allow the supra-thermal particle distribution function to consistently evolve in the presence of a nearly periodic chorus wave packet, we have

$$\frac{4\pi i}{\omega_k}\frac{\delta \boldsymbol{J}_h\cdot\delta\boldsymbol{E}_\perp^*}{|\delta\boldsymbol{E}_{\perp 0k}(z,t)|^2} = -\frac{\omega_p^2}{n\omega_k}\int_{-\infty}^{+\infty} e^{-i\omega t}\left\langle \frac{\mu B}{\Omega+kv_\parallel-\omega_k-i(\gamma_k-i\omega)} \left.\frac{\partial \hat{f}_0(\omega)}{\partial\mathscr{E}}\right|_{\mathscr{C}}\right\rangle_v d\omega \,.$$

Here, $\langle\ldots\rangle_v = \int d\boldsymbol{v}(\ldots)$ and $\partial/\partial\mathscr{E}$ is taken at constant $\mathscr{C} \simeq \mathscr{E} - \mu B_0(\omega_k/\Omega_0)$, which is a non-linear constant of motion. Furthermore, $\hat{f}_0(\omega) = (2\pi)^{-1}\int_0^\infty e^{i\omega t} f_0(t)dt$ is the Laplace transform of $f_0(t)$ and satisfies the following Dyson-like equation

$$\begin{aligned}\hat{f}_0(\omega) &= \frac{i}{2\pi\omega} f_0 - \frac{2i}{\omega}\frac{e^2}{m^2}(2\mu B)^{1/2} e^{-2\gamma_k t}|\delta\boldsymbol{E}_{\perp 0k}|^2 \\ &\times \left.\frac{\partial}{\partial\mathscr{E}}\right|_{\mathscr{C}}\left[\frac{(2\mu B)^{1/2}(\gamma_k+i\omega)}{(\Omega+kv_\parallel-\omega_k)^2+(\gamma_k+i\omega)^2}\left.\frac{\partial}{\partial\mathscr{E}}\right|_{\mathscr{C}}\hat{f}_0(\omega-2i\gamma_k)\right] \,,\end{aligned} \tag{8}$$

which provides the renormalized solution for $\hat{f}_0(\omega)$ in the presence of a nearly coherent chorus wave packet [3]. Equation (8) can be cast into the following evolution equation for γ_k

$$\begin{aligned}\frac{\partial \gamma_k}{\partial t} &= \frac{\omega_p^2}{n\omega_k}\mathbb{I}\mathrm{m}\left\langle \int_{-\infty}^{\infty} d\omega \frac{e^{-i\omega t}\mu B}{\Omega + kv_\parallel - \omega_k - i(3\gamma_k - i\omega)} \frac{\partial}{\partial \mathscr{E}}\bigg|_{\mathscr{C}} \left\{ \frac{2e^2}{m^2}(2\mu B)^{1/2}|\delta \boldsymbol{E}_{\perp 0k}|^2 \right.\right. \\ &\left.\left. \times \frac{\partial}{\partial \mathscr{E}}\bigg|_{\mathscr{C}} \left[\frac{(2\mu B)^{1/2}(\gamma_k - i\omega)}{(\Omega + kv_\parallel - \omega_k)^2 + (\gamma_k - i\omega)^2} \frac{\partial}{\partial \mathscr{E}}\bigg|_{\mathscr{C}} \hat{f}_0(\omega) \right] \right\} \right\rangle_v . \end{aligned} \tag{9}$$

Equations (5) and (9) form a close system of nonlinear equations that can be adopted in the analysis of chorus emission, which is ongoing. Here, to illustrate the implications of Eq. (9), we assume that $(x^2+a^2)^{-1}(x^2+b^2)^{-1} \simeq (\pi/ab)/(a+b)\delta(x)$ for $\mathbb{R}\mathrm{e}a > 0$, $\mathbb{R}\mathrm{e}b > 0$ and $|a|, |b| \ll 1$. In this way, by direct substitution into Eq. (9), we obtain

$$\frac{\partial^2 \gamma_k}{\partial t^2} \simeq \frac{\partial}{\partial v_{\parallel,R}} \left(\frac{\omega_k^2 \omega_{tr}^4}{k^4 v_{\parallel,R}^2} \frac{\partial}{\partial v_{\parallel,R}} \gamma_k \right) , \tag{10}$$

where $v_{\parallel,R} = (\omega_k - \Omega_0)/k$ is the resonant speed and $\omega_{tr}^2 = (e/m)(k^2/\omega_k)(2\mu B)^{1/2}|\delta \boldsymbol{E}_{\perp 0k}|$ is the wave-particle trapping frequency. This result suggests that the fastest growing mode, maximizing wave-particle power exchange, is characterized by non-adiabatic nonlinear dynamics [1, 2] with frequency sweeping

$$\frac{\partial \omega_k}{\partial t} \simeq \frac{\omega_k}{\Omega_0 - \omega_k} \frac{\omega_{tr}^2}{(1 - v_{\parallel,R}/v_g)^2} , \quad \Leftarrow \quad \frac{d}{dt} v_{\parallel,R} \simeq \frac{\omega_k}{k^2 |v_{\parallel,R}|} \omega_{tr}^2 . \tag{11}$$

Despite the drastic simplification in Eq. (10), this result is consistent with the former analysis by Vomvoridis et al. [6] and Omura et al. [7] ($R \simeq 1/2$); and yields the estimate

$$R \equiv (1 - v_{\parallel,R}/v_g)^2 \frac{\partial \omega_k/\partial t}{\omega_{tr}^2} \simeq \frac{\omega_k}{\Omega_0 - \omega_k} . \tag{12}$$

References

[1] X. Tao, L. Chen and F. Zonca, *Some theoretical and numerical studies of chorus generation*. Presented at the AGU Fall Meeting, San Francisco, California, December 14-18 (2015).

[2] X. Tao, F. Zonca and L. Chen, *Identify the nonlinear wave-particle interaction regime during chorus wave generation*, Geophys. Res. Lett. **44**, doi:10.1002/2017GL072624 (2017). Also presented at the 58th Annual Meeting of the APS Division of Plasma Physics - San Jose, California, October 31st - November 4th (2016), Bull. Am. Phys. Soc. **61**, No. 18 (2016).

[3] L. Chen and F. Zonca, Rev. Mod. Phys. **88**, 015008 (2016).

[4] X. Tao, J. Geophys. Res. Space Physics **119**, 3362 (2014).

[5] J. M. Albert, Phys. Fluids **5**, 2744 (1993).

[6] J. L. Vomvoridis, T. L. Crystal, and J. Denavit, J. Geophys. Res. **87**, 1473 (1982).

[7] Y. Omura, Y. Katoh, and D. Summers, J. Geophys. Res. **113**, A04223 (2008).