BIFURCATION OF GRAD-SHAFRANOV EQUATION SOLUTIONS.

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The Grad-Shafranov (GS) equation is the second order nonlinear two-dimensional partial differential equation. This equation give us possibility to calculate an axial-symmetric equilibrium magnetic configurations of a toroidal plasma column with arbitrary boundary cross-section (TOKAMAK type systems). As it is known a nonlinear problem may not have a solution at all, can have one solution or can have more than one solution [1]. Among these solutions there can be classical bifurcation ones, that is, those that describe the qualitative or topological rearrangement of a system that occurs when a certain bifurcation (critical) parameter passes through a bifurcation (critical) point. At the present time, there is no consistent theory of bifurcations for the GS equation.

Numerical solutions of the GS equation show that there are three types of bifurcation solutions of the plasma column equilibrium in a tokamak with non-circular magnetic surfaces.

1. A simple bifurcation in which there is no macroscopic change in the shape of the nested magnetic surfaces $\psi(x,y) = const$ (Fig.1a).

2. Complex bifurcation at which macroscopic distortions of magnetic surfaces occur, but they remain nested (simply connected) [2] (Fig.1b).

3. Topological bifurcation at which additional magnetic axes appear in a tokamak, the system of magnetic surfaces becomes multiply connected and currents along the magnetic axes can

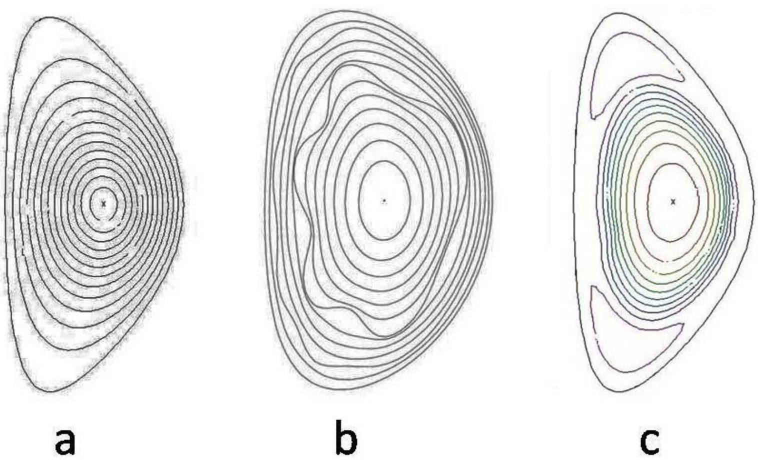

Fig.1. Different types of bifurcations

a - is a simple bifurcation, b- is a complex bifurcation, c- is a topological bifurcation flow in one direction or in opposite directions [3] (Fig.1c).

Bifurcations also include the formation of islands [4], filaments and barriers [5]. It should be noted that bifurcations are observed only for certain values of the parameters entering into the GS equation.

In this study three examples of simple bifurcation solutions of the GS equation in a plasma with a fixed boundary ($\psi_b = 0$) described by the expression [6] are given

$$y_b(x) = \pm \frac{K\sqrt{1-\delta^2}}{1+\delta x}\sqrt{1-x^2} \qquad (1)$$

where K is elongation and δ is triangularity.

1. The GS equation with exponential nonlinearities [7].

The equation of the GS corresponding to a state of a plasma with a minimum of energy, which describes the conservation of plasma parameter profiles and which has bifurcation solutions, has the form

$$h\frac{\partial}{\partial x}\frac{1}{h}\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial y^2} = -hj_s(\frac{\psi}{\lambda}) = -h^2C_p\, exp(\frac{5}{4}\frac{\psi}{\lambda}) - C_F\, exp(\frac{\psi}{\lambda}) \qquad (2)$$

here $h = 1 + x/A$, A is aspect ratio, λ is bifurcation parameter.

In the cylindrical approximation, if $C_p = 0$ or $C_F = 0$, equation (1) has an analytic bifurcation solution [7]. Figure 2 shows the dependence of the flux function maximal value ψ_m in a tokamak on the bifurcation parameter magnitude. Calculations are made for a tokamak with A=3, an elongation K=2, triangularity δ=0.5, C_p=1, C_F=2.

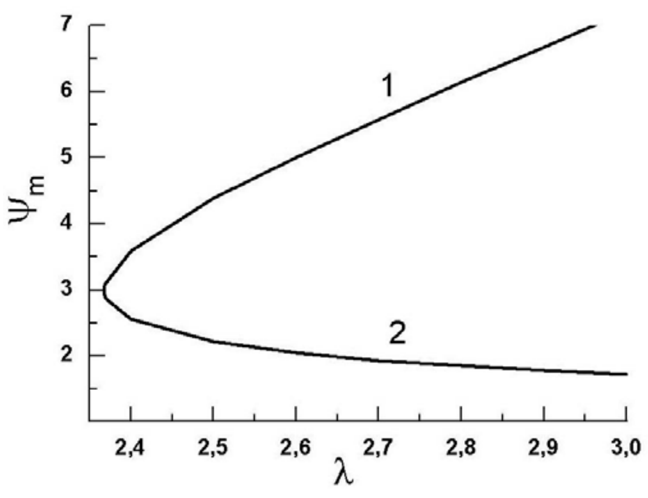

Fig.2. Radial dependence of flux function maximal value ψ_m on bifurcation parameter λ

The upper branch of diagram (1) describes the maximum, and the lower one (2) - the minimum values of ψ_m . As the bifurcation parameter decreases, the flux function maximal value on both branches approach each other and at the bifurcation point λ≈2.4 both solutions coincide. There are no real solutions when λ<2.4.

It can be seen from the figure that in this case a twofold (saddle-node) bifurcation is realized.

Figure 3 shows the radial distribution of the bifurcation current densities calculated

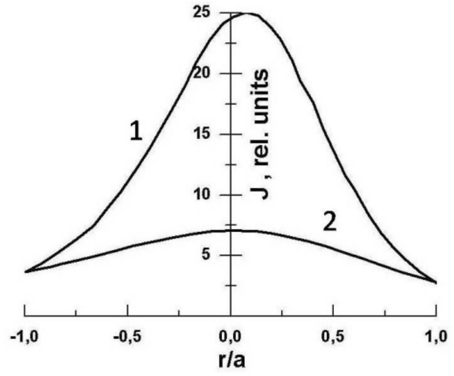

Fig.3. Radial distribution of bifurcation currents.
1 - maximal toroidal current
2 - minimal toroidal current

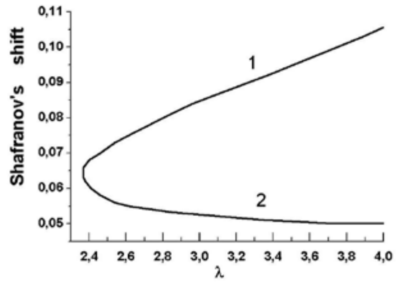

Fig.4. Bifurcation diagram of Shafranov's shift

for λ=2.6. It can be seen from the figure that both current densities differ in magnitude and in radial distribution. The values of the total currents in this case differ by a factor of 1.7. In addition, they have a different Shafranov's shift. The bifurcation diagram for the magnitude of the Shafranov's shift is shown in Fig.4.

Calculations show that the parameters of both bifurcation columns with maximum and minimum currents satisfy the condition corresponding to the pressure balances [8].

2. ITER like tokamak.

The current distribution along the radius in ITER is characterized by the fact that it has a maximum near the plasma boundary [9]. This type of current distribution is described by the next GS equation

$$h\frac{\partial}{\partial x}\frac{1}{h}\frac{\partial \psi}{\partial x}+\frac{\partial^2 \psi}{\partial y^2}=-h^2 D_p f_2(\psi)-D_F f_1(\psi) f_2(\psi) \quad (3)$$

$$f_2(\psi)=0.5+arctg(a_2(\psi-\psi_2))/\pi$$

$$f_1(\psi)=0.5+arctg(a_1(\psi_1-\psi))/\pi,$$

$D_p=1.8,\ D_F=1.8,\ a_1=50,\ a_2=20,$

$\psi_1=0.6,\ \psi_2=0.2$

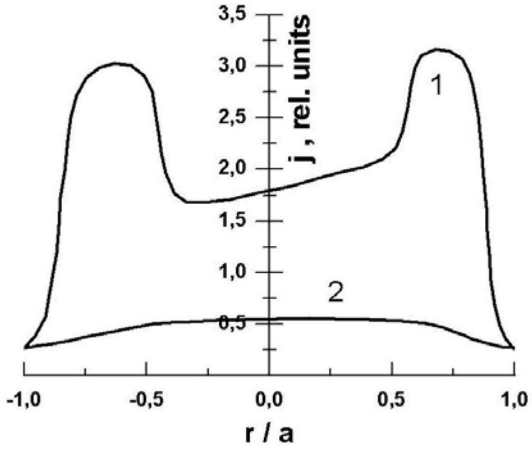

Fig.5. The radial distribution of the bifurcation currents in the ITER-like tokamak

The radial distribution of the bifurcation currents in the ITER like tokamak is shown in Fig.5. Calculations are made for A=3, K=2, δ=0.5.

It can be seen that the radial distributions of the current density in both current columns differ even topologically. The total currents in both cases differ by a factor of 5.3.

3.Tokamak T-15.

The radial distribution of the current density of the T-15 tokamak was calculated by the **A**STRA code. The calculation is performed for a plasma with a total current of 2 MA, $\beta_j = 1.04,\ l_i = 0.57, A = 2.2, K = 1.7, \delta = 0.43$

In this case, the GS equation has the form

$$h\frac{\partial}{\partial x}\frac{1}{h}\frac{\partial\psi}{\partial x} + \frac{\partial^2\psi}{\partial y^2} = -h^2 C_1\psi + C_2(\psi - \psi_c)^2 + C_3 arctg(a_1\psi)/\pi \qquad (4)$$

where $a_1 = 5,\ C_1 = 2,\ C_2 = 1,\ C_3 = 1,\ \psi_c = 0.2.$

The results of the calculations are shown in Fig.6. In this figure, the points denote the quantities found with help of the ASTRA code, solid lines are bifurcational solutions of the GS equation. It can be seen from the figure that the distribution of the maximal current density (curve 1) found for the T-15 tokamak by both methods is almost identical. The minimal value is about 2 times less than the maximal one. The ratio of total currents is 2: 1.

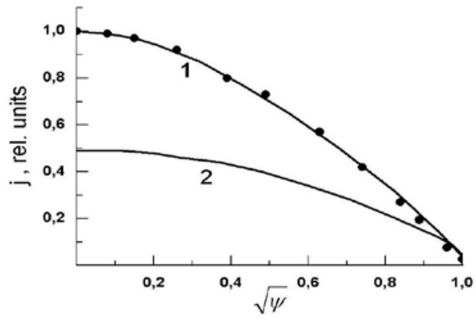

Fig.6. Bifurcation current in T-15

It can be seen from this work that the form of bifurcations depends on the nonlinearity of the GS equation. For example, a stepped profile of the current density leads to the appearance of a three-fold bifurcation [11].

There are no detailed studies of the influence of bifurcation solutions on the plasma equilibrium at the present time. Only the possibilities of the influence of these solutions on the formation of barriers [10] and L-H transitions [7,12] in the tokamak plasma were considered.

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