Banana kinetic equation and plasma transport in tokamaks

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In the core of fusion grade tokamak plasmas, where $\nu_* = \nu Rq/\left(\varepsilon^{3/2}\mathrm{v}_t\right)$ is much less than unity, banana kinetics becomes important for modes with wavelengths comparable to or shorter than the width of bananas, and with frequency ω lower than the gyro-frequency $\Omega = eB/(Mc)$ and the bounce frequency of bananas [1-3]. Here, ν is the typical collision frequency, R is the major radius, q is the safety factor, ε is the inverse aspect ratio, $\mathrm{v}_t = \sqrt{2T/M}$ is the thermal speed, T is the temperature, e is the charge, c is the speed of light, B is the magnetic field strength, and M is the mass. Because the width of the gyro-orbits ρ is assumed to be much less than the wavelengths of the modes, gyro-kinetics is neglected and the drift kinetic equation is used for the banana kinetics.

To treat banana kinetics, we choose $\left(p_\zeta,\theta,\zeta_0,E,\mu,t\right)$ as independent variables [1-3]. Here, p_ζ is the toroidal component of the canonical momentum, θ is the poloidal angle, $\zeta_0 = q\theta - \zeta$ is the field line label, ζ is the toroidal angle, E is the particle energy per unit mass, and μ is the magnetic moment per unit mass. We employ Hamada coordinates [4], in which the equilibrium magnetic field is expressed as $\boldsymbol{B}_0 = \psi' \nabla V \times \nabla\theta - \chi' \nabla V \times \nabla\zeta$, where $\psi' = \boldsymbol{B}_0 \bullet \nabla\zeta$, χ and ψ are respectively the poloidal and toroidal flux divided by 2π, , $\chi' = \boldsymbol{B}_0 \bullet \nabla\theta$, and prime denotes d/dV. The covariant representation for $\boldsymbol{B}_0$ is $\boldsymbol{B}_0 = G\nabla\theta + F\nabla\zeta + \nabla\varphi$, where $F = F(\chi)$, and $G(\chi)$ are the poloidal current outside, and the toroidal current inside a magnetic surface multiplied by $c/2$, respectively. The function φ satisfies the equation $\boldsymbol{B_0} \bullet \nabla\varphi = B_0^2 - \left\langle B_0^2\right\rangle$ [5], where angular brackets denote flux surface average.

We choose the electrostatic and vector potentials to represent perturbed electromagnetic fields. For the electrostatic potential ϕ , $\phi = \phi_0 + \phi_1 = \phi_0(V) + \sum_{lmn}\phi_{lmn}e^{i(m\theta - n\zeta) + ilk_\chi\chi - i\omega t}$, where $\phi_0(V)$ is the equilibrium potential, ϕ_1 is the perturbed potential, ϕ_{imn} is the Fourier amplitude, k_χ is the radial wave vector in terms of χ, and (l,m,n) are respectively the radial, poloidal, and toroidal mode numbers. For $\omega < \omega_b =$

$v_t\sqrt{\varepsilon}/(Rq)$, the bounce frequency of bananas, it is convenient to use ζ_0, and ϕ becomes $\phi = \phi_0(V) + \sum_{ln}\phi_{ln}(\theta)e^{in\zeta_0+ilk_\chi\chi-i\omega t}$. Because ϕ_1 is real, $\phi_{lmn} = \phi^*_{-l-m-n}$, and $\phi_{ln}(\theta) = \phi^*_{-l-n}(\theta)$, where the superscript * indicates the complex conjugate. Similarly, the perturbed vector potential $\boldsymbol{A}$ is expressed either as $\boldsymbol{A} = \sum_{lmn}\boldsymbol{A}_{lmn}e^{i(m\theta-n\zeta)+ilk_\chi\chi-i\omega t}$, or as $\boldsymbol{A} = \sum_{ln}\boldsymbol{A}_{ln}(\theta)e^{in\zeta_0+ilk_\chi\chi-i\omega t}$. The $\boldsymbol{A}$ and its Fourier amplitudes are decomposed as $\boldsymbol{A} = A_\parallel \boldsymbol{n}_0 + \boldsymbol{A}_\perp$, where $\boldsymbol{n}_0 = \boldsymbol{B}_0/|\boldsymbol{B}_0|$, and the subscripts $\parallel$ and $\perp$ indicate the components parallel and perpendicular to $\boldsymbol{B}_0$, respectively. Also, $\boldsymbol{A}_{lmn} = \boldsymbol{A}^*_{-l-m-n}$, and $\boldsymbol{A}_{ln}(\theta) = \boldsymbol{A}^*_{-l-n}(\theta)$.

We derive the banana kinetic equation using the following orderings. We adopt $k_\perp\rho < 1$, where $k_\perp$ is the magnitude of the wave vector perpendicular to $\boldsymbol{B}_0$. For maximum ordering, we choose $k_\chi\rho_p RB_p \sim 1$, with ρ_p, the poloidal gyro-radius, and B_p, the equilibrium poloidal magnetic field strength. We still assume that $\rho_p < L$, with L, the equilibrium radial gradient scale length.

Following the procedure in [1-3], orbit averaging the drift kinetic equation when ω_b is the dominant frequency yields the banana kinetic equation

$$\frac{\partial f_0}{\partial t}+\left\langle \mathbf{v}_d \bullet \nabla\zeta_0\right\rangle_{ob}\frac{\partial f_0}{\partial\zeta_0}+\left\langle\left(\frac{e}{M}\frac{\partial\phi}{\partial t}-\frac{e}{Mc}\mathrm{v}_\parallel\boldsymbol{n}\bullet\frac{\partial\boldsymbol{A}}{\partial t}\right)\right\rangle_{ob}\frac{\partial f_0}{\partial E}+\left\langle\dot{p}_\zeta\right\rangle_{ob}\frac{\partial f_0}{\partial p_\zeta} = \left\langle C(f_0)\right\rangle_{ob}, \quad (1)$$

where $f_0 = f_0(p_\zeta,\zeta_0,E,\mu,t)$, $\mathbf{v}_d = -\mathrm{v}_\parallel\boldsymbol{n}\times\nabla(\mathrm{v}_\parallel/\Omega) + \left[\mathrm{v}_\parallel^2/(B\Omega)\right](\nabla\times\boldsymbol{B})_\perp + (\mathrm{v}_\parallel/\Omega)\boldsymbol{n}\times(\partial\boldsymbol{n}/\partial t)$ is the drift velocity, $\boldsymbol{n} = (\boldsymbol{B}_0+\boldsymbol{B}_1)/B$, $\boldsymbol{B}_1 = \nabla\times\boldsymbol{A}$, $p_\zeta = \chi - F\mathrm{v}_\parallel/\Omega - \nabla V\times\nabla\theta\bullet\boldsymbol{A}$, $\dot{p}_\zeta = (\mathrm{v}_\parallel/\Omega)\left[\partial(\mathrm{v}_\parallel B^2/\Omega)/\partial\zeta_0+\partial(\boldsymbol{B}_0\bullet\boldsymbol{A})/\partial\zeta_0\right]$, the orbit averaging operator is $\langle\bullet\rangle_{ob} = \left[\sum_\sigma\int_{\theta_{t1}}^{\theta_{t2}}d\theta B(\bullet)/(|\mathrm{v}_\parallel|\chi')\right]\Big/\left(2\int_{\theta_{t1}}^{\theta_{t2}}d\theta B/(|\mathrm{v}_\parallel|\chi')\right)$, $\sum_\sigma$ is a sum over σ, σ is the sign of the parallel particle speed $\mathrm{v}_\parallel$, and θ_{t1} and θ_{t2} are two turning points at which $\mathrm{v}_\parallel = 0$. The θ integrals in $\langle\bullet\rangle_{ob}$ are performed holding $(p_\zeta,\zeta_0,E,\mu,t)$ constant. Fast bounce motion of bananas smoothens out the θ variation in f_0. Because $\sqrt{\varepsilon}\rho_p/L < 1$, the difference between p_ζ and V on the equilibrium quantities is neglected. Ordering $\omega\sim\omega_d = \left\langle\mathbf{v}_d\bullet\nabla\zeta_0\right\rangle_{ob}\sim\nu > \omega_{dp_\zeta}$, a reduced banana kinetic equation is obtained, i.e.,

$$\frac{\partial f_{01}}{\partial t}+\left\langle\mathbf{v}_d\bullet\nabla\zeta_0\right\rangle_{ob}\frac{\partial f_{01}}{\partial\zeta_0}-\left\langle C(f_{01})\right\rangle_{ob} = -\left\langle\left(\frac{e}{M}\frac{\partial\phi}{\partial t}-\frac{e}{Mc}\mathrm{v}_\parallel\boldsymbol{n}\bullet\frac{\partial\boldsymbol{A}}{\partial t}\right)\right\rangle_{ob}\frac{\partial f_{00}}{\partial E}-\left\langle\dot{p}_\zeta\right\rangle_{ob}\frac{\partial f_{00}}{\chi'\partial V}, \quad (2)$$

where $f_{00} = N\left(\pi^{3/2}\mathrm{v}_t^3\right)^{-1}e^{-(ME/T-e\phi_0/T)}$ is a Maxwellian distribution, and N is density. Equation (2) is used to calculate the fluxes in all the low collisionality regimes as outlined in the theory for neoclassical toroidal plasma viscosity [6]. The explicit expression for the right side of Eq.(2) is

$$-\left\langle\left(\frac{e}{M}\frac{\partial\phi}{\partial t}-\frac{e}{Mc}\mathrm{v}_{\parallel}\boldsymbol{n}\bullet\frac{\partial\boldsymbol{A}}{\partial t}\right)\right\rangle_{ob}\frac{\partial f_{00}}{\partial E}-\left\langle\dot{p}_{\zeta}\right\rangle_{ob}\frac{\partial f_{00}}{\chi'\partial V}$$

$$=\frac{cf_M}{\chi}\sum_{l,n}in\left[-\frac{\omega}{n}\frac{e\chi'}{cT}+\frac{p'}{p}+\frac{e\phi_0'}{T}+\left(x^2-\frac{5}{2}\right)\frac{T''}{T}\right]\left\langle\phi_{ln}-\frac{\mathrm{v}_{\parallel}}{c}A_{\parallel,ln}\right\rangle_{ob}e^{i(lk_{\chi}p_{\zeta}+n\zeta_0-\omega t)}, \quad (3)$$

where p is plasma pressure, $x = \mathrm{v}/\mathrm{v}_t$, and the consequences of the banana kinetics can be found in the orbit averaged potentials

$$\left\langle\phi_{ln}-\frac{\mathrm{v}_{\parallel}}{c}A_{\parallel,ln}\right\rangle_{ob}=\left(\int_{\theta_{t1}}^{\theta_{t2}}d\theta\frac{B}{|\mathrm{v}_{\parallel}|\chi'}\right)^{-1}$$

$$\times\int_{\theta_{t1}}^{\theta_{t2}}d\theta\frac{B}{|\mathrm{v}_{\parallel}|\chi'}\left[\phi_{ln}(\theta)\cos\left(\frac{lk_{\chi}F\mathrm{v}}{\Omega}\frac{\mathrm{v}_{\parallel}}{\mathrm{v}}\right)-i\frac{|\mathrm{v}_{\parallel}|}{\mathrm{v}}\frac{\mathrm{v}}{\mathrm{v}_t}\frac{\mathrm{v}_t}{c}A_{\parallel,ln}(\theta)\sin\left(\frac{lk_{\chi}F\mathrm{v}}{\Omega}\frac{|\mathrm{v}_{\parallel}|}{\mathrm{v}}\right)\right]. \quad (4)$$

It is obvious that bananas sample the perturbed fields along their trajectories. We emphasize that the contribution of the vector potential to the banana kinetic equation remains finite unless the width of bananas vanishes.

We solve Eq.(2) in the superbanana plateau regime caused by the drift resonance,

$$\omega - n\omega_d = 0. \quad (5)$$

The ensemble averaged particle flux $\langle\Gamma\bullet\nabla V\rangle_{en}$ and heat flux $\langle\boldsymbol{q}\bullet\nabla V\rangle_{en}/T$ are, for j = 1- 3,

$$\begin{pmatrix}\langle\Gamma\bullet\nabla V\rangle_{en}\\ \langle\boldsymbol{q}\bullet\nabla V\rangle_{en}/T\end{pmatrix}=-\frac{2|e|c}{\sqrt{\pi}M\chi}\frac{N\mathrm{v}_t^{-2}}{|\varepsilon'|}\sqrt{B_M/B_m-1}\left[\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\left(-\frac{\omega}{n}\frac{e\chi''}{cT}+\frac{p'}{p}+\frac{e\Phi'}{T}\right)+\begin{pmatrix}\eta_2\\ \eta_3\end{pmatrix}\frac{T'}{T}\right], \quad (6)$$

and $\eta_j = \sum_{l,n}\int_{x_{\min}}^{\infty}dx\left(x^2-5/2\right)^{j-1}e^{-x^2}\left[1+k_r^2\left(B_M/B_m-1\right)\right]^{-3/2}|n|\left|\left\langle\phi_{ln}-\mathrm{v}_{\parallel}A_{\parallel,ln}/c\right\rangle_{ob}\right|_{k_r}^2\left|\partial G_k/\partial k_r^2\right|^{-1}\times$ $\int_{\theta_{t1}}^{\theta_{t2}}d\theta\left(B/B_m\right)\left[k_r^2-\left(B/B_m-1\right)/\left(B_M/B_m-1\right)\right]^{-1/2}$. The resonance function $G_k(k) = \lambda\times$ $\int_{\theta_{t1}}^{\theta_{t2}}d\theta\frac{B/\left(B_m\varepsilon'\right)}{\sqrt{k^2-\left(B/B_m-1\right)/\left(B_M/B_m-1\right)}}\left[2\left(\frac{B_M}{B_m}-1\right)\left(k^2-\frac{B/B_m-1}{B_M/B_m-1}\right)\left(\frac{\chi''}{\chi}-\left(B^2\right)'/\left(2B^2\right)\right)+\right.$ $\left.\frac{B}{B_m}\left(B^2\right)'/\left(2B^2\right)\right]\left\{\int_{\theta_L}^{\theta_U}d\theta\frac{B/B_m}{\sqrt{k^2-\left(B/B_m-1\right)/\left(B_M/B_m-1\right)}}\right\}^{-1}$, where $\frac{\chi''}{\chi}=\frac{\langle B^2\rangle'}{\langle B^2\rangle}+\frac{4\pi P'}{\langle B^2\rangle}-F$ $q'\frac{\chi'}{\langle B^2\rangle}$, $\lambda = \mu B_m/E_k$, $E_k = \mathrm{v}^2/2$ is the kinetic energy of the particle per unit mass, P is total plasma pressure, and B_M and B_m are respectively the maximum and minimum values of B

along the banana bouncing trajectories [7]. The trapping state is characterized by the pitch angle parameter $k^2 = (1-\lambda)/[\lambda(B_M/B_m - 1)]$: for trapped particles, $1 \geq k^2 \geq 0$ and for circulating particles, $k^2 \geq 1$. The resonance pitch angle k_r^2 is the solution of the drift resonance condition, i.e., Eq.(5), for a given energy. The lower limit of the energy integral $x_{\min}$ depends on the sign of $G_k(k)$; only those particles with the normalized energy larger than $x_{\min}$ can resonate. In the region $G(k) > 0$, $x_{\min} = \left|[2e\chi/(Mc\mathrm{v}_t^2\varepsilon')](\omega/n - c\Phi_0'/\chi)\{Max[G_k(k)]\}^{-1}\right|^{1/2}$, and in the region $G_k(k) < 0$, $x_{\min} = \left|[2e\chi/(Mc\mathrm{v}_t^2\varepsilon')](\omega/n - c\Phi_0'/\chi)\{Min[G_k(k)]\}^{-1}\right|^{1/2}$. The *Max* and *Min* are defined respectively as the maximum and minimum values of the argument.

The transport fluxes in Eq.(6) can be used to model transport behaviors of thermal particles and energetic alpha particles in the presence of electrostatic turbulence, Alfvenic waves, and chaotic magnetic fields when magnetic field correlation length $L_M > Rq$ in tokamaks. For electrostatic turbulence, we set $\boldsymbol{A} = 0$, and banana kinetics is stabilizing or improves confinement. For Alfvenic waves, it is the interference between the electrostatic and vector potentials that determines if banana kinetics improves or degrades plasma confinement. In chaotic magnetic fields, we set $\phi_1 = 0$ and $\omega = 0$, and when $k_\chi \rho_p R B_p \sqrt{\varepsilon} \sim 1$, drift resonance enhances transport losses over transit or bounce frequency resonance induced losses, e.g., Rechester-Rosenbluth coefficient [8], by a factor of $L/\rho_p > 1$. The banana kinetics can be used to selectively pump out undesirable species.

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