Certain developments on the equilibrium of magnetized plasmas

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Introduction: It has been established in various fusion devices that sheared flows play an important role in the transitions to improved confinement regimes such as the L-H transition and the formation of internal transport barriers. These flows can be driven externally with either electromagnetic waves or neutral beam injection employed for plasma heating and current drive, or can be created spontaneously (intrinsic flows). Another important effect of external sources, depending on the direction of the injected momentum, is pressure anisotropy [1, 2], which owing to small collission frequency in high temperature plasmas is sustained for long time, thus affecting the confinement properties. Also, in large devices as ITER two-fluid effects are expected to become noticeable. In the present contribution recent results will be presented on steady states of magnetically confined plasmas obtained by conventional and Hamiltonian methods. The presentation consists of three parts. The first one concerns the derivation of a generalized Grad-Shafranov (GGS) equation describing helically symmetric equilibria with pressure anisotropy and incompressible flow of arbitrary direction with application to straight-stellarator configurations [3]. The impact of pressure anisotropy and flow on the equilibrium characteristics is also examined. In the second part the axisymmetric equilibrium code HELENA is extended for pressure anisotropy and flow parallel to the magnetic field. In the third part the Hamiltonian formulation of helically symmetric plasmas is established within the framework of extended MHD (XMHD), a simplified two-fluid model [4, 5].

Generalized Grad-Shafranov equation with anisotropic pressure and flow: The ideal magnetohydrodynamic equilibrium states with plasma flow and anisotropic pressure are governed by the following system of equations

$$\rho(\mathbf{v}\cdot\nabla)\mathbf{v} = \frac{1}{\mu_0}(\nabla\times\mathbf{B})\times\mathbf{B} - \nabla\cdot\mathbb{P},\ \ \nabla\times(\mathbf{v}\times\mathbf{B}) = 0,\ \ \nabla\cdot(\rho\mathbf{v}) = 0,\ \ \nabla\cdot\mathbf{B} = 0\,. \tag{1}$$

Here $\mathbb{P} \equiv p_\perp\mathbb{I} + \sigma_d\mathbf{BB}/\mu_0$ is the CGL pressure tensor [6], where the function $\sigma_d \equiv \mu_0(p_\parallel - p_\perp)/B^2$ measures pressure anisotropy. For incompressible flow, under the assumption that σ_d is uniform on magnetic surfaces and the condition of helical symmetry, we have derived a GGS equation in helical coordinates $(r,\, u = m\phi - kz,\, z)$, where $(r,\, \phi\, z)$ are cylindrical coordinates [3]:

$$\mathscr{L}U + \frac{2kmqX}{(1-\sigma_d - M_p^2)^{1/2}} + \frac{1}{2}\left(\frac{X^2}{1-\sigma_d-M_p^2}\right)' + \frac{\mu_0}{q}\bar{p}_s' + \frac{\mu_0}{2q^2}\left[(1-\sigma_d)\rho(\Phi')^2\right]' = 0\,. \tag{2}$$

Here $U(r,u) = \int_0^\psi (1-\sigma_d(f) - M_p^2(f))^{1/2}df$, where $(2\pi m/k)\psi$ is the poloidal magnetic flux; $\rho(U)$ is the mass density, $X(U)$ is related to the helical magnetic field, M_p is the poloidal Mach function related to the parallel component of the flow, $\Phi(U)$ is the electrostatic potential in connection with the non-parallel component of the flow; $\bar{p}_s(U)$ is the static part of an effective pressure defined as $\bar{p} \equiv (p_\perp + p_\parallel)/2$; $q \equiv (k^2r^2 + m^2)^{-1}$ is a scale factor connected with the helical symmetry; $\mathscr{L} \equiv (1/q)\nabla\cdot(q\nabla)$; and the prime denotes derivative with respect to U. The equilibrium is governed by (2) and a Bernoulli equation for $\bar{p}$ [3]. As an example, in Fig. 1 we present 2-dimensional equilibria of Wendelstein 7x, depicting features in the limit of zero toroidicity and constant torsion, constructed by an analytic solution of Eq. (2).

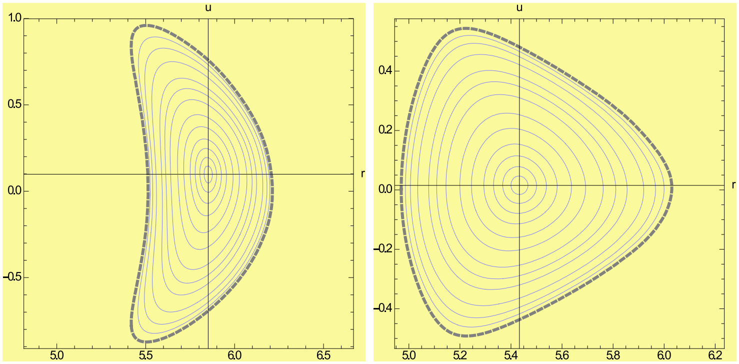

Figure 1: *Poloidal cuts of the magnetic surface which remain invariant along the helical direction resembling respective ones of WD7-x stellarator for toroidal angles* 0^o *and* 36^o *[7]-[8].*

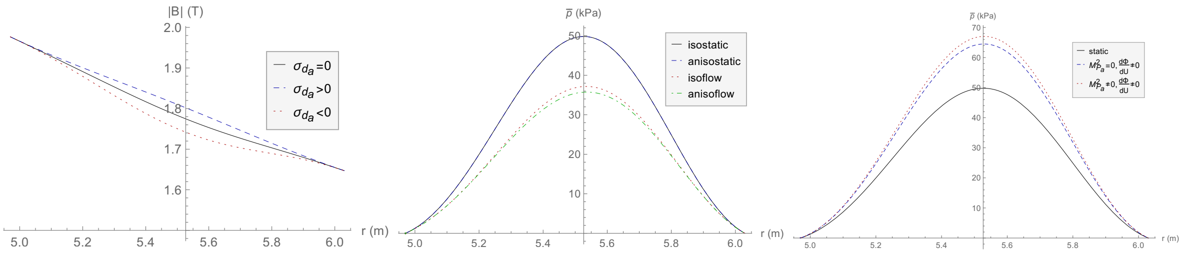

Figure 2: *The left figure shows the paramagnetic/diamagnetic impact of anisotropy. The central figure illustrates the synergetic paramagnetic action of parallel flow and anisotropy in the absence of electric field. The right figure shows the diamagnetic impact of the electric field, which is enhanced by the parallel flow.*

Also, we found that pressure anisotropy can act either paramagnetically (for $\sigma_d > 0$) or diamagnetically (for $\sigma_d < 0$); note that σ_d can be either positive or negative depending on the direction of the auxiliary heating. For $\sigma_d > 0$ magnetic-field-aligned flow has an additive paramagnetic impact to that of anisotropy. The non-parallel flow has a diamagnetic impact, which becomes stronger as M_p^2 takes larger values (Fig. 2).

Numerical axisymmetric equilibria with pressure anisotropy and parallel flow: For parallel flow Eq. (2) becomes identical in form with the usual isotropic static Grad-Shafranov equation. The impact of plasma flow can be examined by means of the (total) Alfvénic Mach function, M, which for parallel flow becomes identical with the poloidal Alfvénic Mach function, M_p . Here we have extended the axisymmetric equilibrium code HELENA [9] to equilibria with pressure anisotropy and parallel incompressible flow on the basis of the axisymmetric form of Eq. (2) (for parallel flow). In this case the physical quantities are calculated by means of the aforementioned transformation, $U(r,z) = \int_0^{\psi}(1-\sigma_d(f)-M_p^2(f))^{1/2}df$, in a manner similar to that employed for the extension of HELENA for parallel flow and isotropic pressure [10]. The free functions $\sigma_d(U)$ and $M^2(U)$ which can be peaked on- or off-axis have been chosen respectively as:

$$F = F_0\left(U^m - U_b^m\right)^n, \; F = C\left[\left(\frac{U}{U_b}\right)^m\left(1-\left(\frac{U}{U_b}\right)\right)\right]^n, \tag{3}$$

where F stands for M^2 or σ_d; m, n are shaping parameters; and $C = F_0[(m+n)/m]^m[n/(m+n)]^n$.

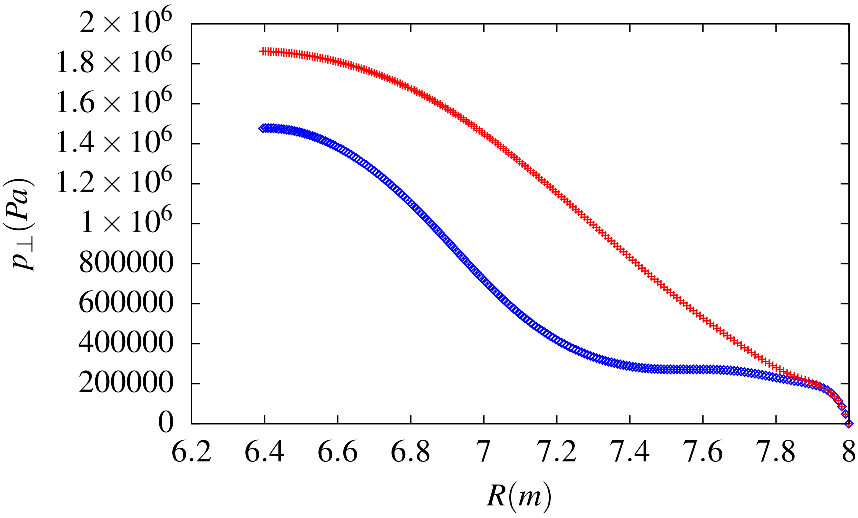

Figure 3: *Perpendicular pressure profiles for* $M_0 = 0.04$, $m_M = 5$, $n_M = 2$, $\sigma_0 = 0.02$, $m_\sigma = 2$, $n_\sigma = 3$ *(red curve); and* $M_0 = 0.04$, $m_M = 5$, $n_M = 2$, $\sigma_0 = -0.02$, $m_\sigma = 2$, $n_\sigma = 3$ *(blue curve).*

The results indicate that the pressure anisotropy affects some quantities such as the current density or the magnetic field while for others, such as the effective pressure, the impact of σ_d is activated only in the presence of flow. In the latter case the impact of σ_d is weak compared to that caused by the flow. The presence of both pressure anisotropy and flow provides freedom in the profile shaping and results in larger values of some equilibrium quantities. For example, in the case of peaked on-axis σ_d and peaked off-axis M^2 the shape of the perpendicular pressure can be changed effectively as can be seen in Fig. 3.

Two-fluid effects: A generalization of the aforementioned equilibrium studies for isotropic pressure can be obtained if in addition to macroscopic flows one considers two-fluid effects. In our study we employ simplified versions of the complete two-fluid model obtained by the imposition of the quasineutrality condition. Such models are the compressible, barotropic ($P = P(\rho)$) XMHD and the Hall MHD (HMHD). The former is obtained by expanding in the smallness of the ratio m_e/m_i and keeping up to first order terms and the latter by assuming massless electrons. The equilibrium studies were conducted within a Hamiltonian framework by constructing the corresponding helically symmetric Hamiltonian formulation, through the computation of the symmetric Hamiltonian functional and noncanonical Poisson bracket [4, 5]. Equilibrium equations were obtained through the Energy-Casimir variational principle. This principle allows for equilibrium and stability studies through the variation of the Hamiltonian functional with constraints being the various Casimir invariants of the model, which are functionals Poisson-commuting with any arbitrary functional F defined in the functional phase space. We applied this procedure for XMHD which is described by the following equations

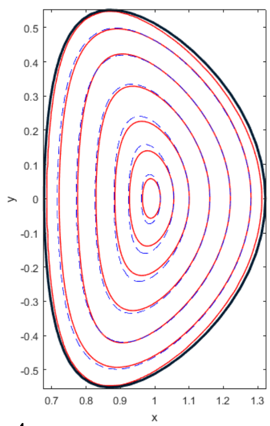

Figure 4: *Ion flow surfaces (solid-red) and magnetic surfaces (dashed-blue) with* $d_i = 0.03$ *for a “straight” Tokamak HMHD equilibrium. The solid black line represents the boundary.*

$$\partial_t\rho = -\nabla\cdot(\rho\mathbf{v}), \quad \partial_t\mathbf{v} = \mathbf{v}\times(\nabla\times\mathbf{v}) - \nabla v^2/2 - \rho^{-1}\nabla p + \rho^{-1}\mathbf{J}\times\mathbf{B}^* - d_e^2\nabla\left(\frac{|\mathbf{J}|^2}{2\rho^2}\right),$$
$$\partial_t B^* = \nabla\times(\mathbf{v}\times\mathbf{B}^*) - d_i\nabla\times\left(\rho^{-1}\mathbf{J}\times\mathbf{B}^*\right) + d_e^2\nabla\times\left[\rho^{-1}\mathbf{J}\times(\nabla\times\mathbf{v})\right], \tag{4}$$

where $\mathbf{J} = \nabla\times\mathbf{B}$, $\mathbf{B}^* = \mathbf{B} + d_e^2\nabla\times(\nabla\times\mathbf{B}/\rho)$ and the parameters d_i and d_e are normalized ion and electron skin depths respectively. The HMHD system is obtained by $d_e = 0$. We found that the symmetric versions of this model possess four families of Casimirs denoted by $\mathscr{C}_i$, $i =$

$1,...,4$. Employing the Energy-Casimir equilibrium variational principle, $\delta(\mathscr{H}-\sum_{i=1}^{4}\mathscr{C}_i)=0$, for the helically symmetric version, we found a system of equilibrium equations which can be cast in the form of a Grad-Shafranov-Bernoulli (GSB) set, consisting of three coupled partial differential equations (PDEs) with respect to the poloidal magnetic flux ψ, and two additional poloidal stream functions $\varphi=\psi^*+\gamma_+ q^{-1/2}v_h$ and $\xi=\psi^*+\gamma_- q^{-1/2}v_h$; here ψ^* is the poloidal flux function of the generalized magnetic field $\mathbf{B}^*$, v_h is the component of the helical velocity, and $\gamma_\pm=[d_i\pm(d_i^2+4d_e^2)^{1/2}]/2$. Those PDEs are additionally coupled to a Bernoulli equation [4, 5]. Setting $d_e=0$ one obtains the Hall MHD GSB system. For reasons of conciseness we present here only the HMHD GSB; the reader is referred to [5] for the complete helically symmetric XMHD system and to [4] for its translationally symmetric counterpart. The HMHD GSB system of equations is

$$d_i^2\mathscr{F}'\nabla\cdot\left(\frac{q}{\rho}\nabla\mathscr{F}\right)=q(\mathscr{F}+\mathscr{G})\mathscr{F}'+\rho\mathscr{M}'-q\left[\frac{\rho}{d_i^2}+2kmq\mathscr{F}'\right](\varphi-\psi)\,, \tag{5}$$

$$\tilde{\mathscr{L}}\psi=q(\mathscr{F}+\mathscr{G})\mathscr{G}'+\rho\mathscr{N}'+2kmq^2(\mathscr{F}+\mathscr{G})+q\rho\frac{(\varphi-\psi)}{d_i^2}\,, \tag{6}$$

$$h(\rho)=\left[\mathscr{M}+\mathscr{N}-q\frac{(\varphi-\psi)^2}{2d_i^2}\right]-d_i^2q\frac{(\mathscr{F}')^2}{2\rho^2}|\nabla\varphi|^2\,, \tag{7}$$

where $\mathscr{F}$, $\mathscr{M}$ and $\mathscr{G}$, $\mathscr{N}$ are arbitrary functions of $\varphi=\psi+d_iq^{-1/2}v_h$ and ψ respectively. The operator $\tilde{\mathscr{L}}$ is $\tilde{\mathscr{L}}\equiv-q\mathscr{L}=-\nabla\cdot(q\nabla)$. The system (5)-(7) is elliptic for subsonic poloidal flows and becomes hyperbolic for $v_p^2>c_s^2$. We computed numerically an HMHD equilibrium state by solving the above system, in the subsonic regime, assuming translational symmetry ($k=0$). The resulting equilibrium configuration, obtained by second-order polynomial ansatzes for the free functions, is depicted in Fig. 4. We observe that the flow surfaces depart from the magnetic ones, as expected in the framework of HMHD model. Determining the separation distance of the two sets of characteristic surfaces may be of interest for transport studies.

Acknowledgments: This work has been carried out within the framework of the EUROfusion Consortium and has received funding from (i) the National Programme for the Controlled Thermonuclear Fusion, Hellenic Republic and (ii) Euratom research and training program 2014-2018 under grant agreement no. 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission. A.E. and D.A.K. were supported by PhD grants from the Hellenic Foundation for Research and Innovation (HFRI) and the General Secretariat for Research and Technology (GSRT). P.J.M. was supported by the US Department of Energy contract DE-FG05- 80ET-53088 and a Forschungspreis from the Alexander von Humboldt Foundation.

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