Complex-eikonal description of geodesic acoustic mode dynamics

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Introduction

The achievement of plasma fusion is strongly related to the understanding of the non-linear mechanisms by which the turbulence self-organizes in convective structures such as streamers and zonal flows [1]. The latter are modulated by, and interact with, a multitude of other structures, instabilities, waves and so on, regulating the energy transport properties in tokamak devices. For example, the zonal flow presents an oscillation counterpart named geodesic acoustic mode (GAM), that in the last decades received much attention for its potential role in the fusion energy confinement. The GAM interacts with turbulence in an environment in which the plasma shape and profile gradients strongly affect its properties. Temperature and density gradients influence the amplitude of GAMs by regulating the drive and the damping in a not completely understood manner. Moreover, the non-uniform temperature profile can determine an evolution of frequency of GAMs, affecting the properties of propagation of these structures [2].

Because of the complexity of these problems, the investigation of zonal flows, GAM dynamics, etc... certainly requires new simulations and new experiments. However, in parallel, it is crucial to develop and to apply new techniques and diagnostics able to capture and to distinguish the essential characteristics of specific mechanisms. For this purpose, in this work we apply techniques derived in the field of optics to the GAM oscillations. In this new context, we show that these theories represent a heuristic base for the description of many characteristics of radial propagation and spreading of GAMs. The most attractive feature of these methods is represented by their universality and intuitive applicability. Moreover, in many cases optical techniques are the only possible approximation for calculating wave fields in the presence of homogeneous and inhomogeneous media. In the following, we present and apply two different optical methods able to describe the spreading of GAMs in terms of local inhomogeneous plane waves. The adopted methods are the complex eikonal theory [3] and the paraxial WKB (pWKB) approximation [4, 5]. Both of them present several advantages. The former is based on a set of partial differential equations and in several cases gives a direct intuitive picture of the physical dynamics of the phenomena. The latter method deals with a set of more simple ordinary differential equation and presents an important advantage from a computational point of view. These methods have been previously applied to the problem of RF heating, current drive and plasma diagnostics with microwave beams in fusion domain. Here, we apply them to GAM dynamics and we demonstrate their efficacy by means of a comparison with gyrokinetic simulations.

Model

In this work, we use the global gyrokinetic code ORB5 [6], which now includes all extensions made in the NEMORB project [7]. The ORB5 code uses a Lagrangian formulation based on the gyrokinetic Vlasov-Maxwell equations. The code solves the full-f gyrokinetic Vlasov equation for ions, using a particle-in-cell δf method. Electrons are treated adiabatically, yet they can also be treated drift-kinetically with ORB5. Energy and momentum conservations can be proved via gyrokinetic field theory. To obtain the potential Φ, the Vlasov equation must be coupled to the quasi-neutrality condition. In the code, time t is normalized to the inverse of the ion cyclotron frequency $\Omega_i = eB_0/m_i$ at a specific radial position. The radial direction is normalized to $\rho_s = \sqrt{k_B T_{e,0} m_i}/(eB_0)$ in which $T_{e,0}$ is the electron temperature, and the potential is given in $\Phi_0 = k_B T_{e,0}/e$ units. The quantity B_0 is calculated in the axial radial position, and $T_{e,0}$ is calculated in the middle of the radial domain. The thermal ion Larmor radius is defined as $\rho_i = \sqrt{2}\sqrt{T_{i,0}/T_{e,0}}\rho_s = \sqrt{2}\sqrt{1/\tau_e}\rho_s$ where $T_{i,0}$ is the ion temperature again in the middle of the radial domain.

Complex-eikonal and Beam tracing methods applied to the GAM evolution

The Complex eikonal theory and the beam tracing equations have been developed to describe inhomogeneous wave fields such as Gaussian beams that propagate in homogeneous or inhomogeneous media. The peculiarity of this approach consists in reducing the Maxwell equations to a set of simpler differential equations taking into account the diffraction effects which are disregarded in the standard geometrical optics. In this way, it becomes possible to separate very efficiently the dispersive effects from those linked to the dissipative properties of the medium. In this section we discuss the complex eikonal approach and the beam tracing methods and we apply them in an original manner to the GAM dynamics. As explained in [3], the complex eikonal description arises from the conventional theory of geometrical optics with the principal equations represented by the eikonal and the transport expression respectively:

$$(\nabla S)^2 - n^2 = 0 \qquad \nabla^2 S + (2\nabla S \cdot \nabla)\ln E = 0 \tag{1}$$

where E is the amplitude of a signal that propagates in a medium with an index of refraction n . In this description, we consider a complex phase $S = s + i\phi$ in which the real part s determines the propagation of the equiphase surfaces, while the imaginary part ϕ describes the phase-path and the attenuation of the signal [3]. For fields with $\phi = 0$, the ray paths of geometrical optics are co-directional with ∇S. They are straight in homogeneous, but curved in inhomogeneous media. By substituting the complex phase S in the eikonal equation and by indicating with $\hat{\boldsymbol{\tau}}$ and $\hat{\mathbf{z}}$ the unit vector tangent to the phase-path and equiphase trajectories respectively, we obtain:

$$\beta^2 - \alpha^2 = n^2 \qquad \beta^2 C = n dn/dz + \alpha d\alpha/dz \tag{2}$$

where $\beta = |\nabla s|$, $\alpha = |\nabla \phi|$ and C is the curvature related to the phase-path. We observe that Eq. 2 takes into account the gradient of the index of refraction n and the gradient of α linked to the beam. In order to apply the eikonal theory, we recall that the n gradient must be smaller than the gradient of the beam packet. When α is constant, we find the classical eikonal solution. It is interesting to observe that even in the case in which we have a constant refraction index n_0, the curvature C can be different from zero (e.g. peripheral rays of a diffracting Gaussian beam). Here, we focus on this case describing a GAM that evolves in a homogeneous equilibrium. For a more general description we refer the reader to Ref. [8]. We consider an initial Gaussian electric field $E(r) = E_0 \exp[-(r-r_0)^2/W_0^2]$ that propagates in a (t,r)-plane (see Fig.1). For small t values the directions $\hat{\boldsymbol{\tau}}$ and $\hat{\mathbf{z}}$ can be approximated to those of the cartesian vectors $\hat{\mathbf{t}}$ and $\hat{\mathbf{r}}$. Thus, by using Eq. 2 we can write the equation $r(t,p)$ for the phase-paths:

$$C \approx d^2r/dt^2 \approx (\alpha d\alpha/dz)/\beta^2 = p(2/n_0W_0^2)^2 \longrightarrow r(t,p) = p[(2t^2/n_0^2W_0^4+1] \tag{3}$$

where p is a parameter that identifies the initial position of a generic phase-path from the center of the packet. In particular we can calculate the spreading of the signal by writing $W(t)$ as a function of the index of refraction n_0 associated to the medium and of the initial width W_0 of the packet:

$$W(t)^2 = W_0^2[1+2t^2/n_0^2W_0^4] \tag{4}$$

in agreement with known results of classical optics [9]. The same result can be obtained by using the pWKB approach. This method also starts from the complex eikonal expression $S = s + i\phi$ but provides a solution of the real and complex part of S by means of an expansion around the reference eikonal ray s_0:

$$s(t) = s_0(t) + n(t)[x_\alpha - \bar{x}_\alpha(t)] + (1/2)s_{\alpha\beta}(t)[x_\alpha - \bar{x}_\alpha(t)][x_\beta - \bar{x}_\beta(t)] + \dots \tag{5}$$

$$\phi(t) = (1/2)\phi_{\alpha\beta}[x_\alpha - \bar{x}_\alpha(t)][x_\beta - \bar{x}_\beta(t)] + \dots \tag{6}$$

Thus, all time-dependent quantities, such as the coordinates and the index of refraction $n(t)$ are calculated along the reference ray $s_0(t)$. In particular, along this ray we note that $\phi = 0$ by construction. Moreover, we have that $s_{\alpha\beta}$ is a function of the wave front curvature and $\phi_{\alpha\beta} = 2/W(t)^2$ is related to the width W of the Gaussian beam. Because of the Hamilton-Jacobi nature of the eikonal equation these two functions can be written in terms of the Hamiltonian expression H associates to the system. For the one dimensional problem of GAMs that oscillate in a homogeneous equilibrium we recall that the dispersion relation is $\omega^2 = \omega_0^2(1+\alpha_1 k_r^2 \rho_i^2)$ [2]. The equation for the evolution of the beam envelop $\bar{s} = s_{\alpha\alpha} + i\phi_{\alpha\alpha}$ can be written in the form:

$$d\bar{s}/dt = -(\partial^2 H/\partial k^2)\bar{s}^2 \quad with \quad H = \omega_0^2(1+\alpha_1 k_r^2\rho_i^2) - \omega^2 = 0 \tag{7}$$

We assume no initial focusing and the initial condition $\phi_0 = 2/W_0^2$. Then, the solution for the evolving width of the packet is:

$$\phi_{\alpha\alpha}(t) = \phi_0/(1+\omega_0^2\alpha_1^2\rho_i^4\phi_0^2t^2) \longrightarrow W^2(t) = W_0^2[1+(2\omega_0\alpha_1\rho_i^2 t/W_0^2)^2] \tag{8}$$

We observe that Eq. 4 and Eq.8 are the same and we can associate an index of refraction n_0 to the equilibrium conditions for the GAM evolution. In order to verify this relation we performed

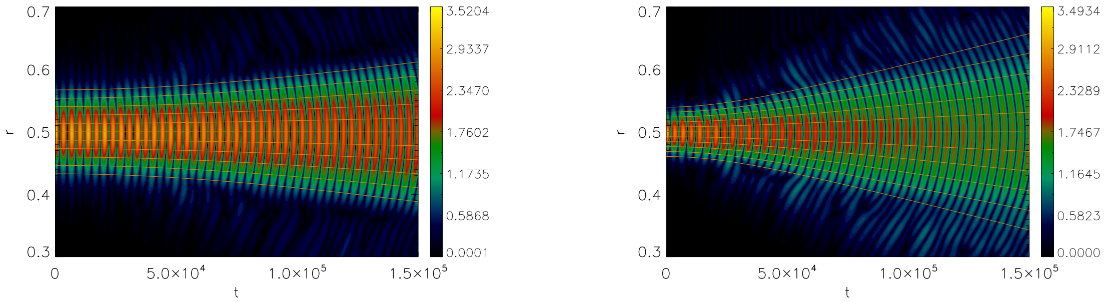

Figure 1: *Time evolution of GAMs with Gaussian electric field profiles oscillating in a homogeneous equilibrium for two different values of* W_0*. The overlapped phase-path trajectories (orange lines), predicted analytically, well reproduce the spreading of GAMs.*

several simulations with the gyrokinetic code ORB5. We assumed homogeneous conditions of temperature and density, an inverse aspect ratio $\varepsilon = 0.1$, a diameter $L_r = 2/\rho^* = 320$ with $\rho^* = \rho_s/a$. We considered a regime in which the dissipative effects, such as the Landau damping, are small in order to emphasize the dispersive effects described by the theory. This has been achieved by considering a safety factor $q = 3$ and large τ_e values [2]. The results are shown in Fig. 1 in which we present the time evolution of two GAM Gaussian profiles with two different width values $W_0 = 0.04$ (left panel) and $W_0 = 0.02$ (right panel). We observe that the theory well reproduces the spreading of the GAM packet. In particular the spreading increases by decreasing the value of W_0. Moreover, the spreading increases in time in agreement with the value of the index of refraction associated to the equilibrium conditions. Thus, the spreading of GAM in time can be predicted and interpreted as a diffraction effect.

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