Superdiffusive transport in plasma for a finite velocity of carriers: general solution and the problem of automodel solution

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1. Introduction. A wide range of physical problems needs describing the transport in the medium for a finite velocity of carriers. The processes of nonlocal transport which description significantly differs from the conventional diffusion approach are of special interest (see, e.g., the survey [1]). The energy transfer by photons in spectral lines of atoms and ions in plasmas and gases in astrophysical objects, nonlocal heat transport by electromagnetic waves in plasmas, migration of predators belong to such processes. These phenomena have superdiffusive character and have to be described by an integral equation for perturbation density, irreducible to diffusion differential equation.

The phenomenon of superdiffusion is closely related to the concept of Lévy flights [2, 3]. The known example of such phenomenon in plasmas is the radiative transfer in the Biberman-Holstein model [4-6]. This model considers resonance photon scattering by an atom or ion with complete redistribution over frequency in the act of absorption and re-emission. Here, rare distant flights of photons («jumps»), which correspond to emission/absorption in the «wings» of spectral line, dominate over contribution of frequent close displacements which produce diffusive (Brownian) motion and correspond to emission/absorption in the core of spectral line. The distant flights caused by the long-tailed (e.g. power-law) wings of integral operator (of the step-length probability distribution function (PDF)) in the transport equation are called Lévy flights [7].

Analysis of the Green's function of non-stationary Biberman-Holstein equation for radiative transfer in plasmas and gases has shown [8] that there is an approximate automodel solution based on three scaling laws: for the front propagation from the instant point source and for asymptotic solutions far behind and far ahead of the propagation front.

This work is devoted to obtaining the general solution of the Biberman-Holstein-type equation with account of finite propagation velocity of perturbation in plasmas, on the way to check the accuracy of the approximate automodel solution similar to the result [8] which is valid for infinite velocity of carriers.

2. General solution for Green's function. Let's consider a one-dimensional (1D) problem of perturbation transport in the medium by Lévy flights with constant finite velocity «*c*» (such processes are named «Lévy walks», see, e.g., [1]). For example, such a task corresponds to excitation transfer by photons in a rarefied extended medium, e.g., in astrophysical plasmas.

The equation for the Green's function of one-dimensional analogue of the Biberman-Holstein equation in a homogeneous medium, with account of finiteness of photon's velocity, has the form (derivation of this equation is given in [9]):

$$\frac{\partial f(x,t)}{\partial t} = -\left(\frac{1}{\tau}+\sigma\right) f(x,t) + \frac{1}{\tau}\int_{-\infty}^{+\infty} dx' W(|x-x'|) f(x',t-\frac{|x-x'|}{c})\theta(t-\frac{|x-x'|}{c}) + \delta(x)\delta(t), \quad (1)$$

where τ is the lifetime of the excited atomic state with respect to spontaneous radiative decay; σ is the rate of (collisional) quenching of excitation; the last term is an instant point source of excited atoms, different from the absorption of the resonant photons (e.g., collisional excitation); $W(\rho) = \frac{\gamma}{2(1+\rho)^{\gamma+1}}$ is the step-length PDF with respect to the process of photon's emission and subsequent absorption after passing the distance ρ, $\int_{-\infty}^{+\infty} W(|x-x'|)dx' = 1$, $0 < \gamma < 2$; $\theta(y)$ is the Heaviside function; $\delta(y)$ is the Dirac function.

General solution of Eq. (1) in dimensionless form (*t* is measured in τ, *x* is measured in the inverse units of absorption coefficient in the center of spectrum line, κ_0) has the form [9]:

$$f(x,t) = \frac{1}{(2\pi)^2 i}\int_{-\infty}^{+\infty} dp e^{ipx} \lim_{\alpha\to+0}\int_{\alpha-i\infty}^{\alpha+i\infty} \frac{e^{st} ds}{s+1+\sigma\tau-\gamma\int_0^{+\infty}\frac{e^{-su/R_c}\cos(pu)}{(1+u)^{\gamma+1}}du}, \quad (2)$$

where parameter $R_c = c\tau\kappa_0$ is the ratio of the mean perturbation lifetime (for radiative transfer, this is the lifetime of the excited atomic or ionic state) to the mean time of flight (for carriers with a finite velocity). For $R_c \to \infty$, Eq. (1) transforms into 1D equation of the Biberman-Holstein type (see Eq. (1) in [8]), and the general solution Eq. (2) coincides with Eq. (18) in [8].

3. Asymptotics. To build the automodel solution similar to that in [8], one has to know the asymptotic expressions far ahead and far behind the propagation front. Expression for asymptotics far ahead the front was derived in [9]:

$$f(x\to\infty, t-|x|/R_c \to 0) = \left(t-|x|/R_c\right)W(|x|)\theta(t-|x|/R_c). \quad (3)$$

Here we consider the case $\gamma = 0.5$ that corresponds to the case of resonance radiative transfer for Lorentz-type spectral line of atoms and ions in plasmas. For example, it can be realized in the collisional (dynamic) Stark broadening of spectral line by charged particles (see, e.g., in [6] the asymptotics of the Holstein function which describes the probability of free path at long distances). For $R_c/t \to 0$ and $R_c t \to \infty$ (large lifetime of atom/ion excited state is considered) we obtain from Eq. (2) the analytical expression for asymptotics far behind the front, which specifies the numeric coefficient in Eq. (19) in [10], valid for $R_c >> 1$:

$$f(x \to 0, t \to \infty) = \frac{1}{R_c^{1/2} t^{3/2}} \frac{2}{\pi^2} \left[\frac{\pi}{8} + \frac{1}{4} \operatorname{arccth}\left(2\sqrt{2+\sqrt{3}} \right) \right] = \frac{0.0930}{R_c^{1/2} t^{3/2}} \quad (4)$$

Analysis of Eq. (2) has shown that building the approximate automodel solution with a finite velocity requires the asymptotics behind the front in a wider range, namely at $R_c t \to \infty$ and not small R_c/t, where the following expression is obtained ($I = \sqrt{\pi/2}$):

$$f(x \to 0, t \to \infty, R_c) = \frac{1}{R_c^{1/2} t^{3/2}} A\left((R_c/t)^{1/2} \right),$$

$$A\left((R_c/t)^{1/2} \right) = \frac{2}{\pi^2 I} \int_0^{+\infty} dw \int_0^{+\infty} du \frac{\cos(u) \left[|u-w|^{1/2} + |u+w|^{1/2} \right] + \sin(u) \left[\frac{2}{I} \left(\frac{R_c}{t} \right)^{1/2} u + \operatorname{sgn}(u-w)|u-w|^{1/2} + \operatorname{sgn}(u+w)|u+w|^{1/2} \right]}{\left[|u-w|^{1/2} + |u+w|^{1/2} \right]^2 + \left[\frac{2}{I} \left(\frac{R_c}{t} \right)^{1/2} u + \operatorname{sgn}(u-w)|u-w|^{1/2} + \operatorname{sgn}(u+w)|u+w|^{1/2} \right]^2}. \quad (5)$$

Numerical results for asymptotic expressions far behind the front, (4) and (5), for $t = 100$ and $R_c = 1$ appear to differ by a factor ~2: $f_{(4)} = 9.23 \cdot 10^{-5}$ and $f_{(5)} = 18.1 \cdot 10^{-5}$.

4. Exact numerical results. Numerical calculation of the general solution (2) and its comparison with the asymptotics (3)-(5) have shown that spatial distribution of perturbation density in the medium in the case of comparability of the time of flight (for carriers with finite velocity) and the perturbation lifetime can have a non-monotonic profile. The results for $t = 100$ for $R_c = 1$ are shown in Fig. 1. The profile for larger velocities, namely $R_c = 100$, for the same time moment, $t = 100$, in comparison with the approximate self-similar solution for infinite velocity (cf. Eq. (9) in [8]), is presented in Fig. 2.

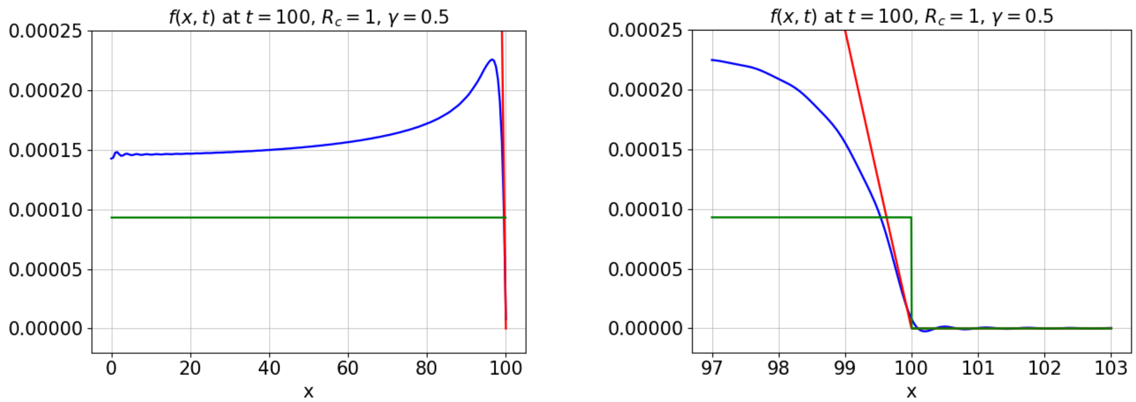

Figure 1. Results of numerical calculation of the exact solution (2) (blue curve), the asymptotics far ahead the front (3) (red curve) and far behind the front (4) (green curve) for t=100 and R_c=1.

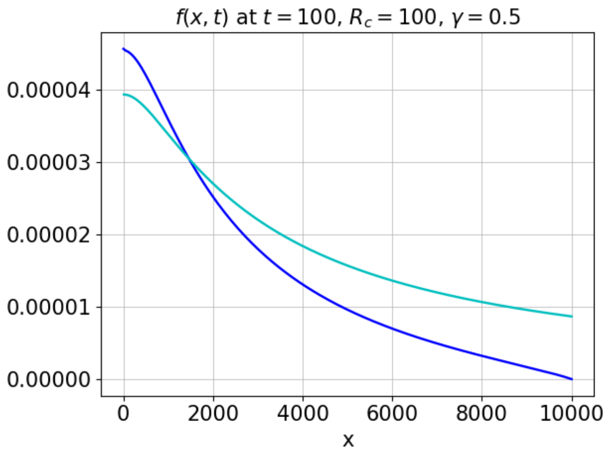

Figure 2. Comparison of numerical calculation of the exact solution (2) for t=100 and R_c=100 (blue curve) with the approximate automodel solution of the problem for infinite velocity of carriers (photons) for t=100, see Eq. (9) in [8] (cyan curve).

5. Conclusion. Allowance for a finite velocity of carriers significantly influences the space-time form of the Green's function for the nonlocal perturbation transport in a homogeneous medium in the regime of the «Lévy walks» which generalize the well-known «Lévy flights» to the case of a finite velocity of carriers. The derived general solution for the Green's function and the results for its asymptotics open the possibility to obtain an approximate automodel solution to generalize the approach [8] to the case of a finite velocity of carriers.

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