

On universal properties of the plasma–sheath transition and large-size sheath structures

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We investigate properties of the plasma–sheath transition in the Bissell and Johnson (B&J) extension [1] of the archetypal Tonks and Langmuir (T&L) collision-free (CF) plasma and sheath equation [2] (assuming a cold ion source) to discharges with Maxwellian sources of arbitrary temperatures. Our work is based on highly accurate numerical and analytic solutions and simulations of the one-dimensional time-independent kinetic equations for the ion and electron velocity distribution functions (VDFs) $f_{i,e}(x, v)$ coupled with the self-consistent electric field. We consider symmetric boundary conditions at two perfectly absorbing co-planar plates, located at positions $x = \pm L$ and characterized by the electric potential $\Phi(\pm L) \equiv \Phi_W$, assuming that starting from the symmetry plane ($x = 0, \Phi = 0$) the electrostatic potential $\Phi(x)$ decreases monotonically in directions positive and negative directions. The basic equations of the problem are

$$v \frac{\partial f_i}{\partial x} + E \frac{\partial f_i}{\partial v} = S(v^2, \varphi), \quad v \frac{\partial f_e}{\partial x} - E \frac{\partial f_e}{\partial v} = 0, \quad n_i - n_e = \varepsilon^2 \frac{d(E^2/2)}{d\varphi}, \quad (1)$$

where the ion and electron densities are defined by $n_{i,e} = \int f_{i,e} dv$ and quantities related to higher velocity moments $\langle v_{i,e}^m \rangle = \int f_{i,e} v^m dv$, are, e.g. the directional velocities $u_{i,e} = \langle v_{i,e}^1 \rangle / n_{i,e}$. The temperatures $T_{i,e}(\Phi) = m_{i,e}(\langle v_{i,e}^2 \rangle - n_{i,e} u_{i,e}^2) / n_{i,e}$, and other quantities of interest read, in their alternative normalized and un-normalized forms:

$$\begin{aligned} \frac{x}{L} &\leftrightarrow x, \quad \frac{n_{i,e}}{n_0} \leftrightarrow n_{i,e}, \quad \frac{v}{c_{s0}} \leftrightarrow u_{i,e}, \quad \frac{T_{i,e}}{T_{e0}} \leftrightarrow T_{i,e}, \quad \frac{e\Phi}{kT_{e0}} \leftrightarrow \Phi = -\varphi, \quad \frac{c_{s0} f_{i,e}}{n_0} \leftrightarrow f_{i,e}, \\ \frac{LE}{kT_e/e} &\leftrightarrow E, \quad \frac{LS(v^2, \varphi)}{n_0} \leftrightarrow S(v^2, \varphi), \quad \varepsilon \equiv \frac{\lambda_D}{L}, \quad \lambda_D^2 = \frac{\varepsilon_0 k T_{e0}}{n_0 e^2}. \end{aligned}$$

Here, $n_0 = n_i(0) = n_e(0)$, and $T_{e,i,0} = T_{e,i}(0)$, are the density and temperature in the center of the discharge, e is the positive elementary charge, k Boltzmann's constant, ε_0 is the vacuum permeability, $c_{s0} \equiv (kT_{e0}/m_i)^{1/2}$ is the cold-ion sound velocity, with $m_{i,e}$ the ion/electron mass, and $E = -d\Phi/dx \leftrightarrow d\varphi/dx$ is the electric field. The ion source $S_i = R n_n n_{e0} e^{\beta\Phi/kT_e} e^{-m_i v^2/2kT_n} / (2\pi k T_n)^{1/2}$ is a function of potential, i.e., $\sim n_e^\beta = e^{\beta\Phi/kT_e}$ with $\beta = 0, 1$ here. The frequency $\nu_i = R n_n = c_{s0}/L_i$ is either due volume ionization or to an external ion source originating from the perpendicular direction, e.g., in the cases when the model is applied to scrape-off-layer [3] (SOL) plasma, with the source temperature T_n different from the self-consistently established ion temperature $T_i(\varphi)$.

Under the above conditions Eqs. (1) can be combined yielding the (un-normalized) energy density balance expression, which we here consider as a special case of the plasma virial theorem [4]: $\mathcal{V}(\Phi) \equiv 2\mathcal{T} - \varepsilon_0 E^2/2 = \text{const}$, with $2\mathcal{T} \equiv \sum(n_{i,e} k T_{i,e} + n_{i,e} m_{i,e} u_{i,e}^2)$ the kinetic and $\varepsilon_0 E^2/2$ electrostatic-pressure contributions. The one-dimensional quantity $\mathcal{V}(\Phi) = \mathcal{V} = \text{const}$

is constant. Assuming Maxwellian electrons ($n_e = e^{-\varphi}$) the virial equation reduces to

$$n_i(T_i + u_i^2) + T_{e0}e^{-\varphi} - \varepsilon^2 \frac{E^2}{2} = T_{i0} + T_{e0}, \quad \text{with} \quad n_i = e^{-\varphi} - \frac{d}{d\varphi} \left(\varepsilon^2 \frac{E^2}{2} \right), \quad (2)$$

with the normalized temperatures $T_e = T_{e0} \equiv 1$ kept explicitly for convenience. Under the condition that $\varepsilon^2 E^2 / 2$ and its first derivative with respect to the potential ($d/d\varphi$, called pseudo-gradient) are negligible. Eq. (2) (with $T_{e0} = 1$) takes the form $u_i^2 + T_i = (T_{i0} + 1)e^{-\varphi} - 1$ so that with the ion temperature known at *any* location/potential φ the local ion velocity can be *instantly* calculated.

For calculating fluxes to the boundary in fluid approaches we need a proper (rather than "*any*") point/potential such that processes within the sheath can be neglected. Our approach to this problem is illustrated in Fig. 1, where we present the behavior of $\varepsilon^2 E^2 / 2 = \int_0^\varphi (\tilde{n}_i - \tilde{n}_e) d\tilde{\varphi}$ which in Ref. [5] has been found to be rather universal for any ion-source temperature as long as ε is sufficiently small. In Fig. 1 we employ the analytic ($\varepsilon = 0$) and numerical (small finite ε) solutions to the CF T&L discharge, i.e., with a zero-temperature Maxwellian ion-source [6], for obtaining $\varepsilon^2 E^2 / 2 = \int_0^\varphi (\tilde{n}_i - \tilde{n}_e) d\tilde{\varphi}$ and its derivatives. The most remarkable quantity turns out to be the charge density pseudo-gradient $d(n_i - n_e) d\varphi$, being always characterized by an inflection point at $\varphi_{PE}(\varepsilon) \approx \varphi_{PE}(0)$ and a maximum at $\varphi_{SE}(\varepsilon = 0) \approx \varphi_{SE}(0)$. Hence, $\varphi_{SE} - \varphi_{PE} \sim 1/3$ is insensitive to both temperature and ε , with the particular values φ_{PE} found and tabulated in Refs. [5, 7] for a wide range of ion-source temperatures. It has been found that for sufficiently small ε the charge density is negligible at φ_{PE} while the electrostatic pressure is negligible at φ_{SE} . Since the value of the charge-density gradient at φ_{SE} is quite insensitive not only to ε but also to ion-source temperature. For these reasons, separate definitions of the plasma edge (at the potential φ_{PE}), the sheath edge (at the potential φ_{SE}), and the plasma-sheath transition region (PST; between them) have been introduced [5].

Solving the B&J problem with finite ε requires finding the electric field as a kernel of the Poisson equation and the ion VDF from the equations

$$\frac{L/L_i}{\sqrt{2\pi T_n}} \int_0^{\varphi_b} \frac{d\varphi' e^{-\beta\varphi'} e^{\frac{\varphi' - \varphi}{2T_n}}}{E(\varphi')} K_0 \left(\frac{|\varphi' - \varphi|}{2T_n} \right) = e^{-\varphi} + \frac{d(\varepsilon^2 E^2 / 2)}{d\varphi} \quad (3)$$

$$f_i(\varphi - y) = \frac{L/L_i}{\sqrt{2\pi T_n}} \int_0^{\varphi_b} \frac{d\varphi' e^{-\beta\varphi'} e^{\frac{\varphi' - \varphi + y}{T_n}}}{E(\varphi')} \frac{e^{\frac{\varphi' - \varphi + y}{T_n}}}{\sqrt{(\varphi' - \varphi + y)}} , \quad (4)$$

where K_0 is the Bessel function of zeroth order, $y = v^2 / 2$, φ_b stands for the boundaries of integration φ_{PE} or φ_W (depending on whether ε is neglected or not), and the eigenvalue L/L_i is tabulated in Ref. [7] for a large number of T_n values. In Fig. 2 we show the results obtained from the above equations (red lines) for $T_n = 3$, together with results obtained from particle-in-cell (PIC)

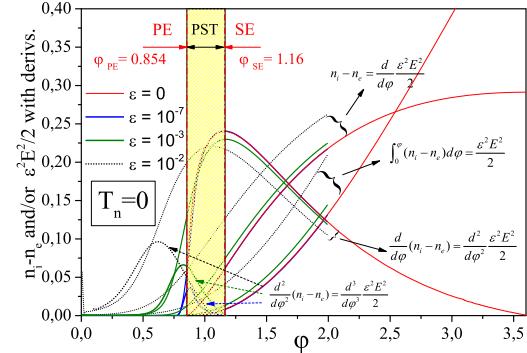


Figure 1: Characteristic points obtained for $T_n = 0$ with several finite ε values.

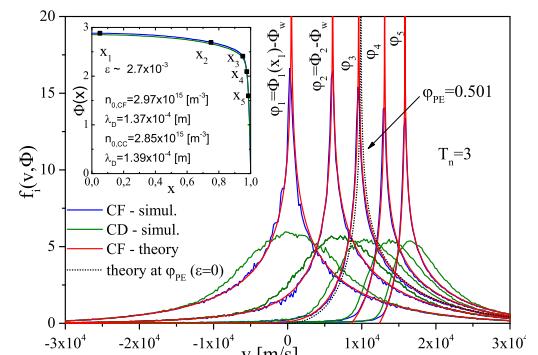


Figure 2: Ion VDFs obtained with PIC simulations in comparison with numerical solutions.

simulations. VDFs from PIC simulations are presented for CF conditions (blue lines) as well as for strongly enhanced ion-ion Coulomb collisions (CC – green curves).

In Fig. 3 we illustrate the simulated ion temperatures T_i obtained with $T_n = 3$ for the "flat" ion source ($\beta = 0$) as functions of potential, together with the electrostatic derivatives. It is evident that the increase in ion temperature, in comparison with the vanishing- ϵ limit, is to considerably increased $\epsilon = 2.7 \times 10^{-3}$ rather than due to Coulomb collisions, i.e. to the particular shape of the ion VDF. Visually, $\epsilon^2 E^2/2 = \int_0^\varphi (\tilde{n}_i - \tilde{n}_e) d\tilde{\varphi}$ and its derivatives behave similarly as those presented for the cold-source discharge presented in Fig. 1, but actually at PE these quantities in the cold (T&L), warm (B&J) and fluid models scale differently. In our approach, relying on the product ϵE (being always a finite quantity), the scaling is

$$\begin{bmatrix} E_{PE} & [\epsilon^2 E^2/2]_{PE} & [n_i - n_e]_{PE} & [d(n_i - n_e)/d\varphi]_{PE} \\ \text{T\&L: } \epsilon^{-4/9} & \epsilon^{10/9} & \epsilon^{6/9} & \epsilon^{2/9} \\ \text{B\&J: } \epsilon^{-2/7} & \epsilon^{10/7} & \epsilon^{6/7} & \epsilon^{2/7} \\ \text{fluid: } \epsilon^{-2/5} & \epsilon^{6/5} & \epsilon^{4/5} & \epsilon^{2/5} \end{bmatrix}.$$

From Fig. 2 and the above scaling one may conclude that $\epsilon^2 E^2/2$ and $(n_i - n_e)$ could be neglected in both plasma and PST regions, i.e., that the simplified expression $u_i^2 + T_i = (T_{i0} + 1)e^{-\varphi} - 1$ might be used for calculating the directional ion sound velocity for any realistically small ϵ . In the sheath region, the full expression 2 must be used. However, it seems that the hypothesis of insensitivity of the relevant terms $\epsilon^2 E^2/2$, $(n_i - n_e)$ and $d(n_i - n_e)/d\varphi$ to particular values of the parameters T_n , ϵ , β , and thus the possible existence of the "universal sheath-wall asymptote" (cf., Fig. 7 and Eq. (30) in Ref. [5]), can be extended to plasmas with Coulomb collisions as well. Here we just illustrate the validity of theoretical Eq. (2) in Fig. 4 corresponding to $T_n = 3$, while simulated results can be found in Ref. [7].

The above results, obviously enable straightforward calculation of the ion directional velocity inside plasma and PST regions at any point, but one might be especially interested in the particular location at which the ion directional ion velocity u_i^2 equals the ion sound speed $c_s^2(\varphi) = T_e^* + \varkappa_i T_i(\varphi)$, where $T_e^* \equiv -n_e/(dn_e/d\varphi) = 1$ for Maxwellian electrons while the differential ion polytropic coefficient function (DPCF) $\varkappa = 1 + (d \ln T)/(d \ln n)$ is tabulated in Ref. [5]. For finite ϵ , the ion directional velocity in the plasma and PST regions can also be represented in the form of the unified Bohm relation

$$u_i^2 \simeq T_e^* + \varkappa_i T_i - \frac{2T_e^* \Gamma_e}{n_e^2} \mathcal{K} + \frac{T_e^{*2}}{n_e} \frac{d^2(\epsilon^2 E^2/2)}{d\varphi^2} \equiv c_s^2 - u_{i,\mathcal{K}}^2 + u_E^2 \quad , \text{ where the term} \\ \mathcal{K}(\varphi) = f_i(\varphi, 0) - 2 \frac{\partial}{\partial \varphi} \int_{-\infty}^{\varphi} dy' f_i^-(y', \varphi), \text{ with } f_i^- = \int_{\varphi}^{\varphi_W} \frac{1}{\epsilon E(\varphi')} \frac{\epsilon S(\varphi' - \varphi + y, \varphi')}{\sqrt{2(\varphi' - \varphi + y)}} \quad , \quad (5)$$

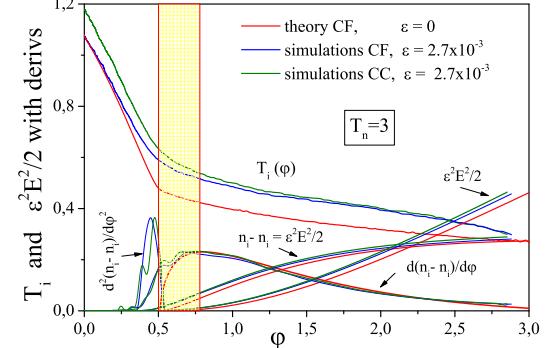


Figure 3: Theoretical ($\epsilon = 0$) and simulated (with finite ϵ) temperatures and density/pressure profiles.

Figure 4: The terms entering the virial obtained for $T_n = 3$ with $\epsilon = 0$.

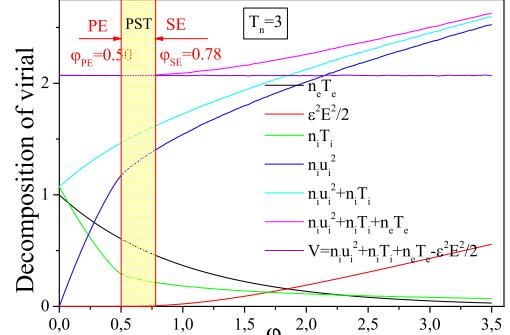


Figure 4: The terms entering the virial obtained for $T_n = 3$ with $\epsilon = 0$.

describes the contribution of the ions originating from the symmetric part of the ion VDF, i.e., those ions which are created between the point of observation and the wall, with the point of observation included. At the sonic point the unified Bohm relation/criterion (5) ($u_i^2 = c_s^2 - u_{i,\mathcal{K}}^2 + u_E^2$) reduces to *two* expressions ($u_i^2 = c_s^2$ and $u_{i,\mathcal{K}}^2 = u_E^2$) with the physical meaning that at the sonic point the rate of ion production is compensated by their removal due to the electric field. We now investigate via PIC simulations the scenario $T_n = T_e = 1$ and $\beta = 0$ (strong source within the sheath) in the manner described in Ref. [8]. In Fig. 5 we present the ion VDFs at several locations/potentials for a relatively high and an extraordinary small plasma density, i.e, with a rather thin and extremely thick sheath, respectively. It can be seen immediately that in the latter case the ion VDF, even when obtained very deep in the sheath (e.g., number 4), is characterized inside a long "tail" of ions which penetrate back into the plasma region. In Fig. 6 profiles of temperatures, polytropic coefficients, directional and ion-sound velocities are plotted showing that for ε_1 and ε_2 the locations of φ_{PE} and φ_{SE} are rather insensitive to ε and the source temperature. This holds relatively well even for a rather high value of ε , such as $\varepsilon_3 = 7.86 \times 10^{-3}$. However, while the sonic point still shifts towards the wall, as in the case of high ion temperatures and $\beta = 1$ in Ref. [5], this shift is rather insignificant here and the value of the ion directional velocity at that point (in contrast to discharges with high source temperatures and $\beta = 1$ from Ref. [5]) here *decreases* with increasing ε . From the physical point of view it should be noted that the ion directional velocity in the case of extremely low density ($\varepsilon_4 = 2.67 \times 10^{-2}$) decreases at *any* point of the discharge, meaning that the "tail" of ions produced in the sheath region has a strong effect to u_i^2 . Nevertheless, as a final result the location of the sonic potential φ_B stays "safely" within the PST region, i.e., not far from $\varphi_{PE} = 0.625$. We conclude that, e.g., $u_{iB}^2 = c_s^2 \approx 1.5$ obtained from the theoretical model with $\varepsilon = 0$, can be regarded as universal for any realistic collision-free discharge with ion source temperature $T_n = T_e = 1$.

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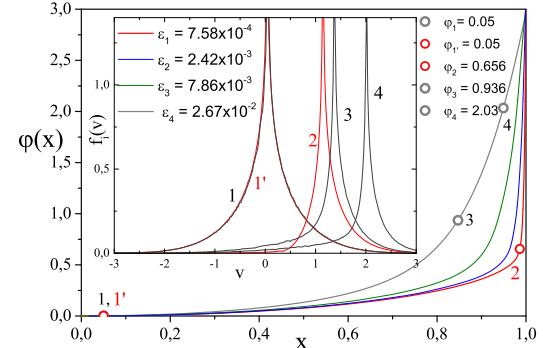


Figure 5: Simulated potential profiles and ion VDFs for several ε values with $T_n = T_e = 1$.

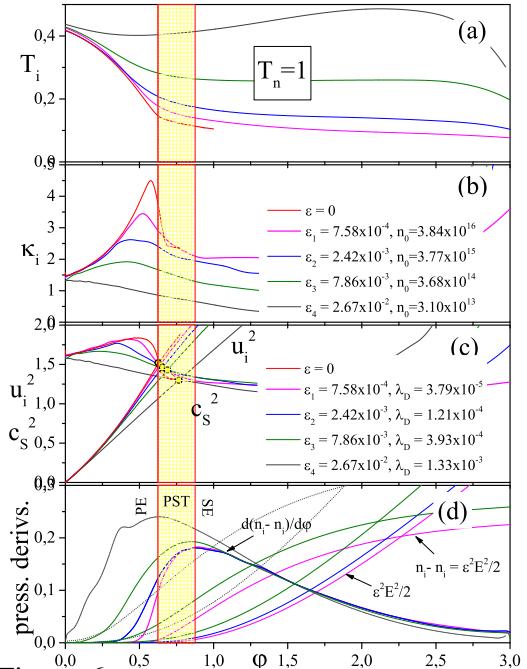


Figure 6: Simulated ion temperature (a), DPCF κ_i (b), u_i^2 and c_s^2 (c), $\varepsilon^2 E^2 / 2$ and $(n_i - n_e)$ with derivatives (d), for $T_n = T_e = 1$.