Weak drift wave turbulence and the statistics of random matrices

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This is an attempt to develop a new approch to renormalization of the particle response to drift wave turbulence, using statistics of the density of eigenvalues of random matrices

The ensemble of drift waves, with random amplitudes and phases (*i.e.* drift turbulence) produces a scattering of the orbit of a particle. The propagator (inverse of the operator of derivation along the particle's trajectory) and the vertex (the nonlinear coupling of drift modes) are modified by terms that must be determined from statistical average over the turbulent fluctuations.

The renormalization can be made by acting upon the propagator by adding a term of diffusion in k space [1]. A more systematic approach uses the diagrammatic expansion of the partition function with the action functional determined according to Martin-Siggia-Rosen method [2]. This will modify the propagator and the vertex. A method that can be used in practical applications is Direct Interaction Approximation (Kraichnan, Terry-Diamond [3]) where the higher order correlations are approximated by assuming a strict channel of transfer of energy by mode coupling. We try here to explore an alternative approach to the renormalization in the drift wave turbulence.

Individually, each "event" (i.e. trajectory of a charged particle in the electrostatic wave) from the statistical ensemble that we invoke when we perform averaging, can be seen either as the unmodified particle's equation of motion but in a modified potential or as a modified equation of motion in the un-changed potential. The statistical averaging does not relay on these distinctions and discerning between these alternatives is simply irrelevant. We only retain from the second alternative that one can see, at least qualitatively, the renormalization as the result of a certain change operated upon the equation of the wave. Being a sort of "pull-back" from the statistical averaging, one cannot say unequivocally what would be that modification.

However it may be useful to explore a class of changes of the drift wave equation that can be seen as inducing renormalization. This is equivalent to look for changes on the equation of drift wave eigenfunctions. An approximative idea about the form of the "modified" eigenfunction that would reflect the renormalization is the expected property of them to be more concentrated in space. This is compatible with the idea that the correction introduced by the renormalization is however small and the scattering of the particle orbits is not too large since this would produce, statistically, contributions to higher order of correlations.

We start by recalling that the set of eigenfunctions of the drift wave instability consists of

Hermite polynomials, $h_k(x)$, $k = 1, N$.

$$h_k(x) = (-1)^k \exp\left(\frac{N}{2}x^2\right)\frac{d^k}{dx^k}\exp\left(-\frac{N}{2}x^2\right)$$

One uses the *monic polynomials* that have coefficient 1 to the highest power of the variable x, $\pi_k(x) = \frac{h_k(x)}{N^k} = \prod_{i=1}^{k}(x - \overline{x}_i)$. Consider a set of hermitean $N \times N$ matrices, H, whose complex entries are random variables. The *elements of the matrices* are stochastic processes in time δt, $H_{ij} \to H_{ij} + \delta H_{ij}$, $\langle \delta H_{ij} \rangle = 0$, $\left\langle (\delta H_{ij})^2 \right\rangle = (1 + \delta_{ij})\,\delta t$. We denote the (random) eigenvalues of these matrices $\{x_i\}_{i=1,N}$. One defines the *joint probability density* $P(x_1, x_2, ..., x_N, t)$. The equation for P is [4]

$$\frac{\partial P}{\partial t} = \frac{1}{2}\sum_{i=1}^{N}\frac{\partial^2 P}{\partial x_i^2} - \sum_{i=1}^{N}\frac{\partial}{\partial x_i}\left[E(x_i)P\right]$$

The connection between the *entries* H_{ij} of the random matrices to the *eigenvalues* is a change of variables with the Jacobian

$$\Delta = \prod_{i=1}^{N}\prod_{j=1,i<j}^{N}\left(x_j - x_i\right)^2$$

The equation shows that there is a Coulomb force acting between the eigenvalues

$$E(x_j) = \sum_{i=1,i\neq j}^{N}\frac{1}{x_j - x_i}$$

The average density of the eigenvalues $\widetilde{\rho}(x,t)$ is obtained from P by integrating over the variables,

$$\widetilde{\rho}(x,t) = \int \prod_{k=1}^{N} dx_k P(x_1, x_2, ..., x_N, t)\sum_{l=1}^{N}\delta(x - x_l)$$

and including the constraint of normalization $\int dx \widetilde{\rho}(x,t) = N$. It is also introduced the scaled density $\widetilde{\rho}(x,t) = N\rho(x,\tau)$ and the function of density in two points

$$\widetilde{\rho}(x,y,t) = \left\langle \sum_{l=1}^{N}\sum_{j=1,j\neq l}^{N}\delta(x - x_l)\,\delta(y - x_j)\right\rangle$$

Using the equation for the joint probability P and in the definition of $\widetilde{\rho}$ one obtains

$$\frac{\partial \rho(x,\tau)}{\partial \tau} + \frac{\partial}{\partial x}\rho(x,\tau)\ \mathrm{P}\int dy \frac{\rho(y,\tau)}{x-y} = \frac{1}{2N}\frac{\partial^2 \rho(x,\tau)}{\partial x^2} + \frac{1}{N}\mathrm{P}\int dy \frac{\rho_c(x,y,\tau)}{x-y}$$

At the limit $N \to \infty$ the RHS can be neglected. We apply the *Hilbert Transform* to the equation and obtain the equation of the resolvent, the ideal (nonviscous) Burgers

$$\frac{\partial G}{\partial \tau} + G(z,\tau)\frac{\partial}{\partial z}G(z,\tau) = 0$$

where $G(z,\tau) = \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{z-H(\tau)}\right\rangle = \int dy \frac{\rho(y,\tau)}{z-y}$. For $z = x - i\varepsilon$ for $x \in \mathbf{R}$, (1) the imaginary part of $G(z,\tau)$ is the average spectral density $\rho(x,\tau)$; (2) the real part of G is the Hilbert transform of ρ.

Now let us look from a different direction:

After rescaling the *time* variable $t \to \frac{\tau}{N}$ the solution of the equation for the probability is

$$P(x_1, x_2, \ldots, x_N, \tau) = C(\tau) \prod_{i<j} (x_i - x_j)^2 \exp\left(-\sum_{i=1}^{N} \frac{N}{2}\frac{x_i^2}{\tau}\right)$$

where $C(\tau)$ is fixed by the normalization. This should be taken as a suggestion: the polynomials that we need should be orthogonal with the *time-dependent measure* $\exp\left(-\frac{N}{2}\frac{x^2}{\tau}\right)$. They are obtained starting from the Hermite polynomials $h_k(x)$ and making the substitution $N \to N/\tau$. The orthogonality relation is

$$\int_{-\infty}^{\infty} dx \exp\left(-\frac{N}{2}\frac{x^2}{\tau}\right) \pi_n(x,\tau)\, \pi_m(x,\tau) = \delta_{nm} c_n^2$$

where $c_n^2 = n!\sqrt{2\pi}\left(\frac{\tau}{N}\right)^{n+1/2}$. This new functions $\pi_k(x,\tau)$ verify the equation

$$\frac{\partial}{\partial\tau}\pi_k(x,\tau) = -\nu_s \frac{\partial^2}{\partial x^2}\pi_k(x,\tau) \quad \text{where} \quad \nu_s = \frac{1}{2N}$$

This is an equation (1) with *negative* dffusion, and (2) with diffusion coefficient $1/(2N)$ inversely proportional with the number of particles. Further, one applies an inverse *Hopf-Cole* transformation and the functions $\pi_k(x,\tau)$ are transformed into

$$f_k(z,\tau) = 2\nu_s \frac{\partial}{\partial z}\ln \pi_k(z,\tau) = \frac{1}{N}\sum_{i=1}^{k} \frac{1}{z - \overline{x}_i(\tau)}$$

The equation for f_k is

$$\frac{\partial f_k(z,\tau)}{\partial \tau} + f_k(z,\tau)\frac{\partial f_k(z,\tau)}{\partial z} = -\nu_s \frac{\partial^2 f_k(z,\tau)}{\partial z^2}$$

The connection is between f_k and π_k and in particular between f_N and π_N. The later is known to be equal to the statistical average of the characterstic polynomial of the matrix $H(\tau)$, $\langle \det[z - H(\tau)]\rangle = \pi_N(z,\tau)$. In the large N limit the last member $f_N(z,\tau)$ coincides with the average of the resolvent $f_N \to G(z,\tau)$ and the *imaginary part of* G gives the density of the eigenvalues of the random matrices.

Modifying the density of eigenvalues G means to replace the equation for the the Hermite polynomials by addition of a supplementary term.

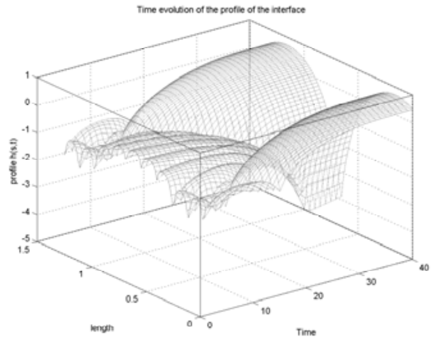

Figure 1: Cusp - strong localization of the Sivashinsky solution.

The suggestion is to concentrate the function f_N in a reduced spatial interval and the modification that seems adequate is to introduce a new term consisting of the Hilbert transform acting on the function f_k, *i.e.* the term $\Lambda[f_k]$. Then the equation becomes the **Sivashinsky** equation for which it is known that the low amplitude fluctuation evolve by coalescence to form a cusp profile. This will provide the expected localization in space.

In **Conclusion**, the connection between drift wave eigenmodes and the density of the eigenvalues of random matrices, which is mediated by the Burgers equation, provides a possibility to implement renormalization by modifying the equation for the electrostatic potential. For more localized functions, the change from Burgers to Sivashinsky equation seems a possible solution.

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References

[1] T. H. Dupree, D.J. Tetreault, Phys. Fluids **21**, 425 (1978).

[2] P.C. Martin, E.D. Siggia, H.A. Rose, Phys.Rev **A8**, 423 (1973).

[3] P. Terry and P.H. Diamond, Phys. Fluids **28**, 1419 (1985).

[4] J.P. Blaizot, M.A. Nowak, Phys. Rev. **E82**, 051115 (2010))