Ray tracing in weakly turbulent, randomly fluctuating media: A quasilinear approach

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Introduction

Ray propagation of electromagnetic and sound waves in turbulent media takes an important role in a varied scope of research areas, ranging from astronomy and free-space communications to the scattering of rf waves in plasmas [1]-[3]. Here, a quasilinear (QL) approach for ray propagation in weakly turbulent media is formulated, which relies on the Hamiltonian form of the ray equations and makes use of a second-order expansion (in the medium and ray fluctuations) of the dispersion relation and ray equations, in order to integrate the ensemble-averaged ray and its root-mean-square (rms) spreading. The QL formalism is validated against Monte Carlo (MC) calculations and, when possible, verified using analytical predictions. For this, a single random mode and a multimode isotropic turbulent spectrum were used as examples.

General quasilinear formalism for ray tracing in random media

The local dispersion relation for a wave with frequency ω propagating in a given medium can be written as:

$$D(\omega,\mathbf{r},\mathbf{k}) \equiv \omega - \omega(\mathbf{r},\mathbf{k}) = 0\,,$$

where $\mathbf{r} \equiv (r_1, r_2, r_3)$ and $\mathbf{k} \equiv (k_1, k_2, k_3)$ are the canonically conjugate coordinate and wave vector. From the above equation it is possible to write the ray equations in their explicit Hamiltonian form:

$$\frac{dr_i}{dt} = \frac{\partial \omega(\mathbf{r},\mathbf{k})}{\partial k_i} \quad \text{and} \quad \frac{dk_i}{dt} = -\frac{\partial \omega(\mathbf{r},\mathbf{k})}{\partial r_i}\,,$$

where t some time-like integration variable along the ray, and there is an $\mathbf{r}$ dependence in $\omega(\mathbf{r},\mathbf{k})$ that enters through the density of the medium $n_e(\mathbf{r})$. The quantities are split into an average $\langle n_e(\mathbf{r})\rangle$, $\langle \mathbf{r}\rangle$ and $\langle \mathbf{k}\rangle$ plus a fluctuating value $\delta n_e(\mathbf{r})$, $\delta\mathbf{r}$, and $\delta\mathbf{k}$, the latter much smaller than the former. The ray equations can then be expanded up to, and including, second-order terms in the fluctuating quantities, yielding expressions for the average coordinates $d\langle r_i\rangle/dt$ and wave vectors $d\langle k_i\rangle/dt$, where (bearing in mind the first order fluctuations vanish by construction) only the zeroth and second order terms on the fluctuations survive. Hence, in what is viewed as a QL approach, the average ray is not the same as the ray one would have if there were no fluctuations. The equations for the evolution of the second-order averaged quantities $\langle \delta r_i \delta r_j\rangle$, $\langle \delta k_i \delta k_j\rangle$,

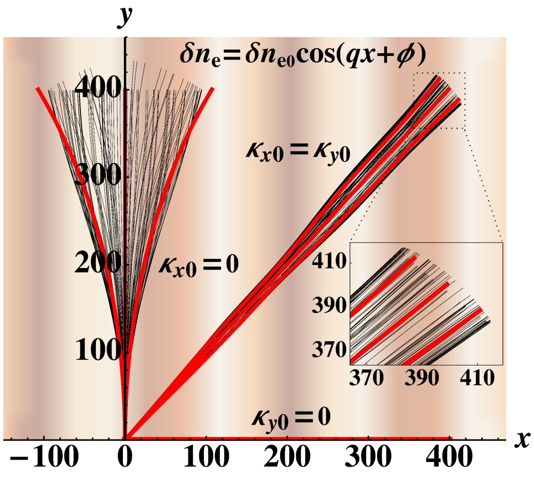

Figure 1: *Average ray trajectories $\langle y \rangle$ vs. $\langle x \rangle$ and their perpendicular rms spreadings $\sigma_\perp$ from the QL formalism (red) vs. a MC calculation with $N = 100$ rays (black) for three distinct initial launching angles (parallel, oblique and perpendicular to the turbulence direction x), superimposed on a single-mode random background. QL results are presented in the form $\langle \mathbf{r} \rangle \pm \sigma_\perp$.*

$\langle \delta r_i \delta k_j \rangle$, $\langle \delta r_i \delta n_e(\langle \mathbf{r} \rangle) \rangle$, $\langle \delta k_i \delta n_e(\langle \mathbf{r} \rangle) \rangle$, $\langle \delta r_i \partial \delta n_e(\langle \mathbf{r} \rangle)/\partial r_j \rangle$, and $\langle \delta k_i \partial \delta n_e(\langle \mathbf{r} \rangle)/\partial r_j \rangle$ can be obtained, whilst ensemble averages such as $\langle \delta n_e(\langle \mathbf{r} \rangle) \delta n_e(\langle \mathbf{r} \rangle) \rangle$ or $\langle \delta n_e(\langle \mathbf{r} \rangle) \partial \delta n_e(\langle \mathbf{r} \rangle)/\partial r_i \rangle$ can be directly computed from the fluctuation model and are evaluated at the average coordinate $\langle \mathbf{r} \rangle$. The implementation of this QL approach gives rise to a formally closed (albeit infinite) set of equations that can be integrated to obtain the expressions for $\langle r_i \rangle$, $\langle k_i \rangle$ and all the second-order terms above. In fact, an infinite recurrence involving higher order derivatives of the density perturbation δn_e appears, which raises the need for some effective truncation in practical implementations (for more insight see [4]).

Rays in homogeneous media with single-mode random fluctuations

The simplest example to consider, also in view of the verification and validation of the QL approach, is that of a single mode propagating with a given wavenumber q along a given direction, say r_1, and with an amplitude δn_{e0}:

$$\delta n_e(\mathbf{r}) \equiv \delta n_e(r_1) \equiv \delta n_{e0} \cos(q r_1 + \phi)\,, \tag{1}$$

where $q \ll k$ (to preserve the consistency with the geometrical optics approach) and ϕ is a random phase distributed uniformly between 0 and 2π. In what follows, x, y, κ_x and κ_y and τ are normalized quantities related to r_1, r_2, k_x, and k_y, respectively, while $\sigma_\perp$ represents the spreading of the ray pencil in the direction perpendicular to the average ray [4].

Results for the ray trajectories launched in a plane with initial directions parallel, perpendicular, and oblique to the direction of the turbulence wave vector are compared in Fig. 1 with MC calculations.

Rays in homogeneous media with isotropic spectra of random fluctuations

An isotropic turbulence spectrum is now considered corresponding to a flat distribution in wave number with $N_q \times N_\theta$ modes, and a prescribed cut-off $q_{\max}$:

$$\delta n_e(\mathbf{r}) \equiv \delta n_e(x,y) \equiv \frac{\delta n_{e0}}{\sqrt{N_q N_\theta}} \sum_{r=1}^{N_q} \sum_{s=1}^{N_\theta} \cos(q_r \cos\theta_s x + q_r \sin\theta_s y + \phi_{rs}),$$

where $q_1 = 0$, $q_{N_q} = q_{\max} \ll k$, $\theta_1 = 0$, $\theta_{N_\theta} = 2\pi$, and the $\phi_{r,s}$ are random phases distributed uniformly between 0 and 2π. In Fig. 2 are shown the distance travelled by the ray and the emsemble-averaged wave vector components for different initial propagation angles. The correspondent ray trajectories and the rms spreadings for two different initial propagation directions are show in Fig. 3, once again compared with the MC results.

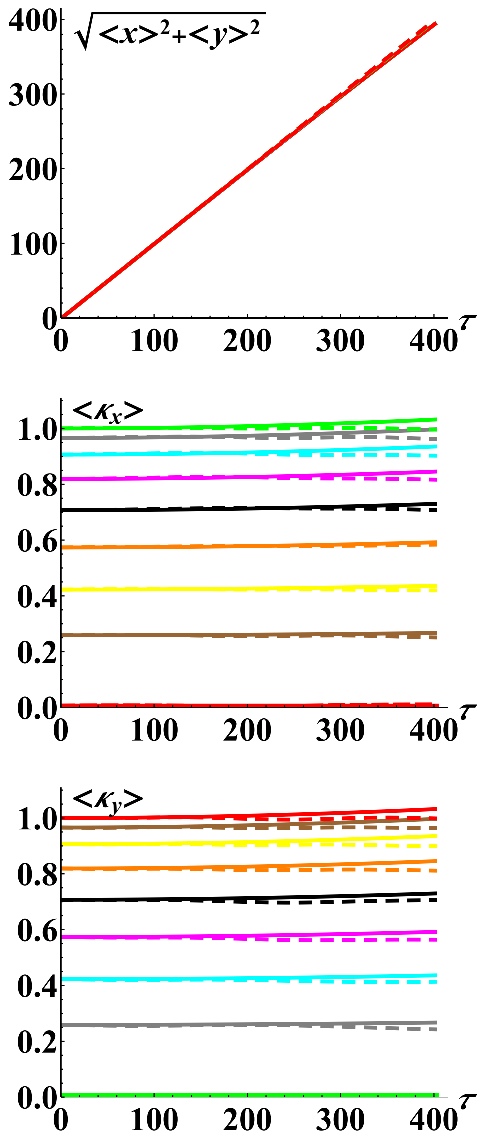

Figure 2: *QL formalism (solid) vs. a MC calculation with $N = 100$ rays (dashed) for $\theta_0 =$ 0 deg (green), $\theta_0 =$ 15 deg (grey), $\theta_0 = 25^o$ (cyan), $\theta_0 = 35^o$ (magenta), $\theta_0 = 45^o$ (black), $\theta_0 = 55^o$ (orange), $\theta_0 = 65^o$ (yellow), $\theta_0 = 75^o$ (brown), and $\theta_0 = 90^o$ (red) and for a $N_q \times N_\theta = 100 \times 100$ multimode isotropic turbulent spectrum: rms travelled distance $\sqrt{\langle x^2\rangle + \langle y^2\rangle}$ and ensemble-averaged wave-vector components $\langle \kappa_x \rangle$ and $\langle \kappa_y \rangle$ as functions of time τ.*

Conclusions

A new QL formalism has been developed to describe the propagation of rays in random media. Keeping second-order terms when emsemble-averaging after expanding the dispersion relation and ray equations leads to a slow drift of the average ray with respect to its unperturbed trajectory. Overall, there is a good agreement between the QL and MC results, particularly for the distance travelled by the average ray, its perpendicular rms spread and the averages of the wave-vector components. This approach comes as an

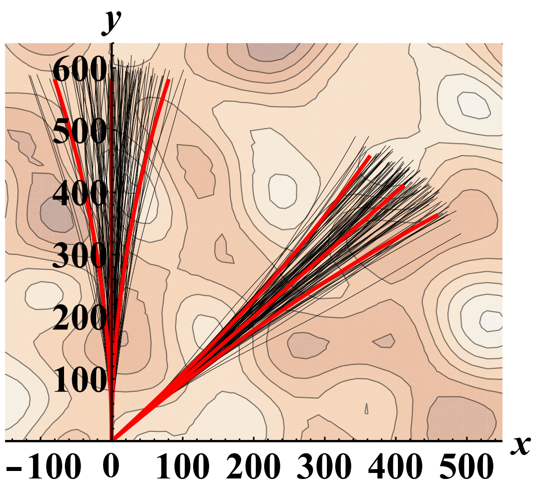

Figure 3: *Average ray trajectories* $\langle y \rangle$ *vs.* $\langle x \rangle$ *and their perpendicular rms spreadings* $\sigma_\perp$ *from the QL formalism (red) vs. a MC calculation with* $N = 100$ *rays (black) superimposed on a* $N_q \times N_\theta = 100 \times 100$ *multimode isotropic random background. QL results are presented in the form* $\langle \mathbf{r} \rangle \pm \sigma_\perp$ *for two distinct initial launching angles.*

efficient alternative to MC calculations and, while similar to the so-called statistical ray tracing [2], it appears to be much easier to implement in the case of more complex geometries or dispersion relations (as when tracing rays in tokamaks). The eventual limitation of the QL formalism arises due to the existence of a downward recursion such that a given order derivative of medium fluctuations depends on the derivatives immediately one order higher. Hence, for practical purposes, the system has to be truncated somehow. Concerning this, it has been shown that not only QL calculations do converge to the correct results when increasing the order of truncation, but also that for a realistic scenario like the one in Fig. 3 convergence happens as early as neglecting fourth order derivatives and higher in the recurrence [4]. The QL formalism showed to be particularly robust on the ensemble average of ray trajectories and the rms width resulting from spreading of the ray pencil in the direction perpendicular to the average ray, and it also showed very good results on the ensemble averages of the wave-vector components.

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