A relativistic Langevin approach for runaway electrons in tokamak plasmas

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1 Introduction

Charged particles in plasmas decrease their Coulomb collision frequency as the energy increases. In this context, electrons with energies higher than some critical threshold are continuously accelerated by the superimposed effects of increasing energy due to the continuous driving of the electric field and decreasing Coulomb collisionality. These electrons are the so-called runaway electrons and, when generated in large amounts, for instance in the case of tokamak disruptions, they could yield serious damage to the structure of the first wall components [1].

In this work, a particle approach to the runaway phenomenon, based on the Langevin equations, including collisional diffusion in momentum space and relativistic effects, has been developed in order to understand better the runaway dynamics. The Langevin equations constitute a technique for studying the motion of charged particles under the stochastic effect of Coulomb collisions with the bulk plasma, allowing to simulate plasma processes in which both collective kinetics effects and Coulomb collisions take place in the dynamics. Particle simulations of plasmas including the stochastic effect of Coulomb collisions by the Monte Carlo treatment or the Langevin technique have been used to address a wide range of problems [2,3]. The Langevin approach is equivalent to the Fokker-Planck treatment, which has been extensively used for the analysis of runaway electron dynamics [4,5]. Synchrotron radiation (SR) losses are also included in the model which: (1) increase the critical (minimum) electric field for runaway generation; (2) set a limit on the maximum energy that the runaways can reach.

In a previous work [6], a similar model was studied, but for non-relativistic electrons. Now, we consider Coulomb collisions between very fast, relativistic electrons and a relatively cold thermal background plasma. The present model is developed using the stochastic equivalence of the Fokker-Plank and Langevin equations [7]. The resulting Langevin equation for relativistic electrons is an stochastic differential equation, amenable to numerical simulations by means of Monte-Carlo type codes.

2 Relativistic Langevin equations

Langevin equations constitute a particle approach for studying the electron motion under the stochastic effect of the collisions with the plasma particles, equivalent to the traditional Fokker-Planck (FP) kinetic approach in the infinite particle limit, but more easily generalized to more complex geometry [3]. The relativistic Fokker-Planck equation for plasma particle species α, colliding with target species β which are practically stationary, can be written as,

$$\frac{\partial f^\alpha}{\partial t} = \frac{\partial}{\partial p_i}\left[-A_i f^\alpha + \frac{1}{2}\frac{\partial}{\partial p_k}\left(B_{ik} f^\alpha\right)\right]. \tag{1}$$

In Eq. (1), A_i and B_{ik} are the Fokker-Planck coefficients (see Ref. [8] for more details),

$$A_i = -\sum_\beta \frac{n_\beta m_\alpha q_\alpha^2 q_\beta^2 \ln\Lambda}{4\pi\varepsilon_0^2}\left(1+\frac{\gamma m_\alpha}{m_\beta}\right)\frac{\gamma p_i}{p^3} = -\sum_\beta \Gamma^{\alpha/\beta}\left(1+\frac{\gamma m_\alpha}{m_\beta}\right)\frac{\gamma p_i}{p^3}, \tag{2a}$$

$$B_{ik} = \sum_\beta \gamma\frac{n_\beta m_\alpha q_\alpha^2 q_\beta^2 \ln\Lambda}{4\pi\varepsilon_0^2}\cdot\frac{p^2\delta_{ik}-p_ip_k}{p^3} = \sum_\beta \Gamma^{\alpha/\beta}\gamma P_{ik}, \tag{2b}$$

where n_j, m_j and q_j are the number density, mass and charge of the specie j respectively, $\ln\Lambda$ is the Coulomb logarithm, $\gamma = (1-v^2/c^2)^{-1/2}$ is the relativistic gamma factor and the tensors $P_{ik} = (p^2\delta_{ik}-p_ip_k)/p^3$, $\Gamma^{\alpha/\beta} = n_\beta m_\alpha q_\alpha^2 q_\beta^2 \ln\Lambda/4\pi\varepsilon_0^2$ have been introduced. The sum in Eqs. (2a)-(2b) is over all species β (including α for self-collisions). This coefficients are only valid, as stated before, when considering Coulomb collisions between very fast, relativistic particles (α) and much slower, background plasma particles (β), which in fact have been taken as practically stationary, greatly simplifying the problem.

The Langevin equation for the particle approach is given by,

$$\frac{dp_i}{dt} = F_i(\vec{p}) + D_{ik}(\vec{p})\xi_k(t), \tag{3}$$

where $\vec{\xi}$ is a gaussian random variable, with zero mean and unit variance. The coefficients A_i and B_{ik} in Eq. (1) are related with F_i and D_{ik} in Eq. (3) (see Ref. [3]) by

$$D_{ij}D_{jk} = B_{ik}; \quad F_i = A_i - \frac{1}{2}D_{jk}\frac{\partial D_{ik}}{\partial p_j}, \tag{4}$$

where the Stratonovich algebra has been used [9]. The resulting Langevin coefficients for the collisions of relativistic electrons with the frozen bulk electrons and ions ($\beta \equiv e, i$) are

$$F_i = -\gamma^2 \sum_\beta \Gamma^{\alpha/\beta}\frac{m_\alpha}{m_\beta}\frac{p_i}{p^3}, \tag{5a}$$

$$D_{ik} = \left[\sum_\beta \Gamma^{\alpha/\beta}\gamma p\right]^{1/2} P_{ik}. \tag{5b}$$

Finally, taking into account these results, the Langevin equation for relativistic runaway electrons, including the force due to the accelerating electric field, $eE_{||}/m_e$, the electron and the ion contributions to the stochastic collision terms, and the electron synchrotron radiation losses, which can be important at relativistic energies, can be written in normalized form,

$$\frac{d\vec{p}}{d\tau} = \vec{D} - \gamma^2\frac{\vec{p}}{p^3} - \frac{\vec{p}\times\vec{p}\times\vec{\bar{\xi}}}{p^2} - \left(\frac{\gamma Z_{\rm eff}}{p}\right)^{1/2}\frac{\vec{p}\times\vec{p}\times\vec{\bar{\eta}}}{p^2} - \left(F_{gc}+F_{gy}\frac{p_\perp^2}{p^4}\right)\gamma^4\left(\frac{v}{c}\right)^3\frac{\vec{p}}{p}. \tag{6}$$

In Eq. (6), $\vec{p}$ is the normalized momentum to the classical momentum at the speed of light, $p_{cl} \equiv m_e c$, and $\tau \equiv t\nu_{ee}$, is the normalized time to the electron collision frequency at the speed of light, with $\nu_{ee} \equiv n_e e^4 \ln\Lambda/4\pi\varepsilon_0^2 m_e^2 c^3$; $\vec{D} \equiv \vec{E}/E_R$ is the normalized electric field, with $E_R \equiv (kT_e/m_ec^2)\,E_D$, and $E_D = (n_e e^3 \ln\Lambda/4\pi\varepsilon_0^2 kT_e)$ is the Dreicer field. Two different gaussian noises, $\vec{\bar{\xi}}$ and $\vec{\bar{\eta}}$, have been considered for the collisions with the bulk electrons and ions respectively, which in turn have also been normalized as $(\vec{\bar{\xi}}, \vec{\bar{\eta}}) = (4\pi\varepsilon_0^2 m_e^2 c^3/n_e q_e^4 \ln\Lambda)^{1/2}(\vec{\xi}, \vec{\eta})$; F_{gc}, F_{gy} are parameters describing the two contributions to the radiation losses [10] (the guiding center motion and the electron gyromotion, respectively), given by $F_{gc} = F_{gy}(m_ec/eB_0R_0)^2$, $F_{gy} = 2\varepsilon_0 B_0^2/3n_e \ln\Lambda m_e$ (R_0 is the plasma major radius and B_0 is the toroidal magnetic field). Finally, $Z_{\rm eff}$ is the effective ion charge.

3 Steady-state solution of the relativistic Fokker-Planck equation without Synchrotron Radiation

The relativistic Langevin equations for runaway electrons can be used to yield the runaway distribution function in momentum space. The initial electron velocities are randomly distributed over a Maxwellian distribution and evolved in time according to Eq. (6). The advanced distribution function is built by a standard statistical method, until a steady-state is achieved. Furthermore, an analytic steady-state solution for the distribution function of runaway electrons, neglecting the radiation losses, can be obtained by imposing the condition of stationarity in the relativistic Fokker-Planck equation, Eq. (1). This steady-state solution is found to be $f^{\alpha} = C\gamma^{-2}$, hence $f^{\alpha}(p) = C/\left(1+p^2\right)$, where C is just a normalization constant. Fig. 1 shows the steady-state distribution functions of runaway electrons obtained in two different ways: 1) via numerical simulations of the relativistic Langevin equation (blue lines), and 2) through the analytical solutions of the corresponding relativistic Fokker-Planck equation (red lines). To appreciate better the agreement between the results of the two models, two

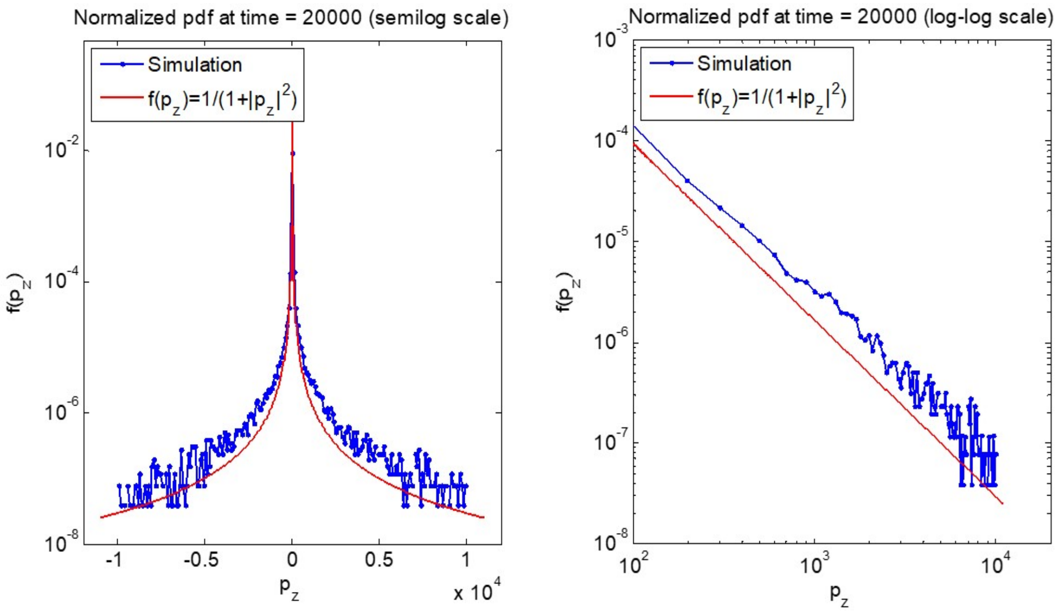

Figure 1: Steady-state probability distribution functions of runaway electrons. (a) Semi-logarithmic scale, (b) log-log scale. In both cases, blue lines represent simulations using the Langevin code and red lines represent the analytic solutions of the Fokker-Planck equation.

different scales have been used, log-lin scale in (a) and log-log scale in (b). The tail for the steady-state distribution of runaway electrons using the Langevin approach in Fig. 1(b) with a decay exponent of 2 is apparent.

4 Critical electric field for runaway generation and runaway energy limits

Relativistic effects yield a critical (minimum) electric field for runaway generation, $D_R = 1$ (normalized to E_R) [11]. The effect of the radiation on the critical field for runaway generation is illustrated in Fig. 2(a) which shows D_R, obtained from the numerical simulations of the relativistic Langevin equation, as a function of the radiation parameter F_{gy}. D_R increases with the strength of the radiation (F_{gy}) and, when the radiation is negligible ($F_{gy} \to 0$), $D_R \to 1$. The results are in agreement with the values provided by a test particle description of the runaway dynamics [10].

Synchrotron radiation also yields a limit on the maximum energy that the runaway electrons can reach. Indeed, as a result of the radiation losses, the generated runaway electrons pile-up

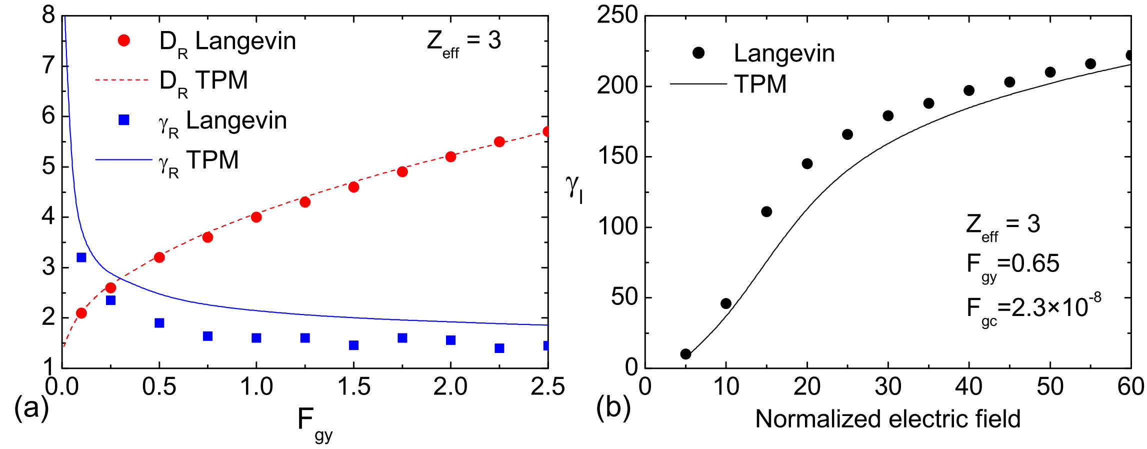

Figure 2: (a) Critical electric field D_R for runaway generation versus F_{gy} (γ_R gives the critical energy for runaway generation at D_R); (b) Runaway energy limit (γ_l) versus the normalized electric field (TPM: Test Particle Model).

at a limiting energy, γ_l, which is shown in Fig. 2(b) as a function of F_{gy}. Such a limiting energy is also consistent with that inferred from the simple Test Particle Model (TPM) of the runaway dynamics.

5 Conclusions

The dynamics of relativistic runaway electrons in tokamak plasmas has been discussed under the Langevin approach to study the random effect of the collisions with the background electrons and ions. The work is based on the formalism of Ref. [3]. The novelty with respect previous works [6] comes from the inclusion of relativistic effects on the runaway electron dynamics as well as the effect of the synchrotron radiation losses.

The steady-state distribution function for runaway electrons has been calculated. The distribution of parallel momentum, relative to the external electric field, decreases as a power law, $f(p_{||}) \sim (1+p^2)$, which has been obtained through both, the numerical simulation of the Langevin equation and the analytic solution of the relativistic Fokker-Planck equation.

The inclusion of the synchrotron radiation losses in the runaway dynamics leads to an increase of the critical electric field for runaway generation ($D_R > 1$), and to a limiting runaway energy, γ_l, both consistent with a simple test particle description of the runaway dynamics [10].

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