# Gyrokinetic theory of toroidal Alfvén eigenmode nonlinear saturation via ion Compton scattering

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Shear Alfvén wave instabilities such as toroidal Alfvén eigenmode (TAE) [1] are expected to play important roles in magnetic confinement fusion devices as energetic particles (EPs) contribute significantly to the total power density [2, 3]. TAE can be driven unstable by EPs, and in turn, induce EP transport and degrade overall plasma confinement. In-depth understanding of TAE nonlinear saturation mechanisms, e.g. ion Compton scattering [4], is thus of importance for the qualitative and quantitative understanding of EP confinement in future tokamaks.

The theory presented in Ref. [4] considered that there exists many TAEs in the system, located at different radial positions with their frequency slightly different by local parameters. TAEs are characterized by $|k_\parallel| \simeq 1/(2qR_0)$. Thus, two counter-propagating TAEs with radially overlapped mode structures can couple and generate ion sound-wave like mode with much lower frequency, and $|k_\parallel| \simeq 1/(qR_0)$; which, in turn, induces the TAE spectral transfer of fluctuation energy towards lower frequency TAEs. The wave energy is, eventually, absorbed by linearly stable lower frequency TAEs with stronger coupling to SAW continuum. The theory of Ref. [4] also assumed the long wavelength MHD limit with $\omega/\Omega_{ci} \gg k_\perp^2\rho_i^2$, where the nonlinear couplings occur through the parallel ponderomotive force induced by $\mathbf{b}\cdot\delta\mathbf{J}\times\delta\mathbf{B}$ nonlinearity. For future burning plasmas of interest, however, the parameters usually fall in the short wavelength $k_\perp^2\rho_{it}^2 > \omega/\Omega_{ci}$ range [3], and the perpendicular coupling due to Reynolds and Maxwell stresses may dominate, leading to much lower TAE saturation level than the prediction of Ref. [4] and consequently, much lower EP transport.

In this work, we generalize the theory of Ref. [4] to fusion plasma relevant short wavelength regime using nonlinear gyrokinetic theory. The analysis, following closely that of Ref. [4], has two major differences; i.e., first, the nonlinear coupling cross-section is much bigger, yielding lower TAE saturation level and EP transport; and second, the scattering cross-section is maximized as the perpendicular wave vectors of the interacting TAEs are perpendicular to each other, which may influence the transport process since TAEs are characterized by $k_r \gg k_\theta$. This second property, contrary to the former, tends to enhance cross-field transport. Since $\omega_{TAE}$ is io the lowest order, independent of $n$, this maximization may determine the $n-$toroidal mode coupling. The details remain to be studied.

## Theoretical model

To investigate the nonlinear TAE spectrum evolution, we adopt the standard nonlinear perturbation theory, and consider a test TAE $\Omega_0 = (\omega_0, \mathbf{k}_0)$ interacting with a background TAE $\Omega_1 = (\omega_1, \mathbf{k}_1)$ and generating a low $n$ ion sound wave like mode $\Omega_S = (\omega_S, \mathbf{k}_S)$. The scalar potential $\delta\phi$ and parallel vector potential $\delta A_\parallel$ are used as the field variables, and one has, $\delta\phi = \delta\phi_0 + \delta\phi_1 + \delta\phi_S$, with the subscripts 0, 1 and $S$ denoting test TAE, background TAE and ion sound mode, respectively. Furthermore, $\delta\psi \equiv \omega\delta A_\parallel/(ck_\parallel)$ is taken as an alternative field variable, and one has $\delta\psi = \delta\phi$ in the ideal MHD limit. Without loss of generality, $\Omega_0 = \Omega_1 + \Omega_S$ is adopted as the frequency/wavenumber matching condition. For effective spectrum transfer by nonlinear ion Landau damping, we have $|\omega_S| \sim O(v_{it}/qR_0)$, i.e., the ion sound mode frequency comparable to thermal ion transit frequency. Therefore, $\Omega_0$ and $\Omega_1$ are counter-propagating TAEs, with $\omega_0 \simeq \omega_1$ and $k_{\parallel,0} \simeq -k_{\parallel,1}$.

The governing equations describing the nonlinear interactions among $\Omega_0$, $\Omega_1$ and $\Omega_S$, can then be derived from quasi-neutrality condition and nonlinear gyrokinetic vorticity equation, while the nonadiabatic particle responses can be derived from nonlinear gyrokinetic equation.

**Nonlinear parametric instability**

The nonlinear generation of ion sound mode due to $\Omega_0$ and $\Omega_1$ beating, is derived from

$$\varepsilon_S \delta\phi_S = i(\hat{\Lambda}/\omega_0)\beta_1 \delta\phi_0 \delta\phi_{1^*}, \tag{1}$$

where $\hat{\Lambda} \equiv (c/B_0)\hat{\mathbf{b}} \cdot \mathbf{k}_0 \times \mathbf{k}_{1^*}$, $\varepsilon_S \equiv 1 + \tau + \tau\Gamma_S \xi_S Z(\xi_S)$ is the linear dispersion function of $\Omega_S$, with $\tau \equiv T_e/T_i$, $\Gamma_S \equiv \langle J_S^2 F_0/n_0\rangle$, $\xi_S \equiv \omega_S/(k_{\parallel,S} v_{it})$ and $Z(\xi_S)$ is the plasma dispersion function. Furthermore, $\beta_1 \equiv \sigma_0\sigma_1 + \tau\hat{F}_1 (1 + \xi_S Z(\xi_S))$, with $\hat{F}_1 \equiv \langle J_0 J_1 J_S F_0/n_0\rangle$, $\sigma_k \equiv 1 + \tau - \tau\Gamma_k$.

Since $\Omega_S$ could be heavily ion Landau damped, one needs to include both its linear and nonlinear responses while deriving the nonlinear particle responses to $\Omega_0$. Substituting it into the quasi-neutrality condition, one has

$$\delta\psi_0 = \left(\sigma_0 + \sigma_0^{(2)}\right)\delta\phi_0 + D_0 \delta\phi_1 \delta\phi_S, \tag{2}$$

in which, $\sigma_0^{(2)} \equiv \hat{\Lambda}^2 \left[-\sigma_1^2\sigma_0 + \tau\hat{F}_2 (1 + \xi_S Z(\xi_S))\right] |\delta\phi_1|^2/\omega_0^2$ and $D_0 \equiv i\hat{\Lambda}\tau\hat{F}_1 \left[1 + \xi_S Z(\xi_S)\right]/\omega_0$. The nonlinear eigenmode equation of $\Omega_0$, can be derived from vorticity equation as

$$\left(\varepsilon_0 + \varepsilon_0^{NL}\right)\delta\phi_0 = -\left(D_2\omega_0^2/\hat{b}_0 + k_{\parallel,0}^2 V_A^2 D_0\right)\delta\phi_1 \delta\phi_S. \tag{3}$$

Here, $\varepsilon_0 \equiv \varepsilon_T(\Omega_0)$ is the WKB linear dispersion relation of $\Omega_0$, with $\varepsilon_T \equiv k_{\parallel,T}^2 V_A^2 \sigma_T - (1 - \Gamma_T)\omega_T^2/\hat{b}_T$, and $\varepsilon_0^{NL} \equiv -\alpha_0^{(2)}/\hat{b}_0 + k_{\parallel,0}^2 V_A^2 \sigma_0^{(2)}$ with $\alpha_0^{(2)} = \hat{\Lambda}^2 \left(\hat{F}_2 - \hat{F}_1\right)(1 + \xi_S Z(\xi_S))|\delta\phi_1|^2$ and $D_2 = -i\hat{\Lambda}\left[\hat{F}_1(1 + \xi_S Z(\xi_S)) - \Gamma_S \xi_S Z(\xi_S) - \Gamma_1\right]/\omega_0$. The TAE eigenmode dispersion relation can then be derived noting the $V_A^2 \propto 1 - 2(r/R_0 + \Delta')\cos\theta$ dependence on poloidal angle $\theta$ with $\Delta'$ being Shafranov shift. $\sigma^{(2)}$ and $\alpha^{(2)}$ correspond, respectively, to the contribution of nonlinear particle response to $\Omega_S$ on ideal MHD constraint breaking and Reynolds stress.

Substituting equation (1) into (3), we obtain

$$\left(\varepsilon_0 + \varepsilon_0^{NL}\right)\delta\phi_0 = -(\hat{\Lambda}^2 \beta_1\beta_2/(\hat{b}_0 \tau\varepsilon_S))|\delta\phi_1|^2 \delta\phi_0, \tag{4}$$

with $\beta_2 \equiv \beta_1/\sigma_0 - \varepsilon_S$. Equation (4) describes the nonlinear evolution of the test TAE $\Omega_0$ due to the nonlinear interactions with $\Omega_1$. Since ion Compton scattering related to $\Omega_S$ ion Landau damping may play an important role for TAE saturation, we write the coefficients explicitly as functions of $\varepsilon_S$, i.e., $\varepsilon_0^{NL} = -(\hat{\Lambda}^2/\hat{b}_0)|\delta\phi_1|^2 \left(\hat{G}_1 + \hat{G}_2\varepsilon_S\right)$ with $\hat{G}_1 = (1 - \Gamma_0)\sigma_1^2 - \sigma_S\hat{G}_2$ and $\hat{G}_2 = \left(\hat{F}_2 - \hat{F}_1 - (1 - \Gamma_0)\tau\hat{F}_2/\sigma_0\right)/(\tau\Gamma_S)$. On the other hand, $\beta_1\beta_2/(\tau\varepsilon_S) = \hat{H}_1 + \hat{H}_2\varepsilon_S + \hat{H}_3/\varepsilon_S$ with $\hat{H}_1 = \left(\sigma_0\sigma_1 - \hat{F}_1\sigma_S/\Gamma_S\right)\left(2\hat{F}_1/\Gamma_S - \sigma_0\right)/(\tau\sigma_0)$, $\hat{H}_2 = \hat{F}_1\left(\hat{F}_1/\Gamma_S - \sigma_0\right)/(\tau\sigma_0\Gamma_S)$, and $\hat{H}_3 =$

$\left(\sigma_0\sigma_1 - \hat{F}_1\sigma_S/\Gamma_S\right)^2/(\tau\sigma_0)$. The nonlinear $\Omega_0$ eigenmode dispersion relation, can then be derived, by multiplying both sides of equation (4) with $\Phi_0^*$, noting that $\varepsilon_S$ varies much slower than $|\Phi_0|^2$ and $|\Phi_1|^2$ in radial direction, and integrating over the radial domain. One then has

$$\varepsilon_S\left(\hat{\varepsilon}_0 - \Delta_0|A_1|^2 - \chi_0\varepsilon_S|A_1|^2\right)A_0 = -\hat{C}_0|A_1|^2A_0, \tag{5}$$

in which $\hat{\varepsilon}_0$ is the linear TAE eigenmode dispersion relation, defined as $\hat{\varepsilon}_0 = \int|\Phi_0|^2\varepsilon_0 dr$. The coefficients, $\Delta_0$, $\chi_0$ and $\hat{C}_0$, corresponding respectively to nonlinear frequency shift, ion Compton scattering and shielded-ion scattering, are given as $\Delta_0 = \langle\langle\hat{\Lambda}^2(\hat{G}_1 - \hat{H}_1)/\hat{b}_0\rangle\rangle$, $\chi_0 = \langle\langle\hat{\Lambda}^2(\hat{G}_2 - \hat{H}_2)/\hat{b}_0\rangle\rangle$, $\hat{C}_0 = \langle\langle\hat{\Lambda}^2\hat{H}_3/\hat{b}_0\rangle\rangle$, with $\langle\langle\cdots\rangle\rangle \equiv \int(\cdots)|\Phi_0|^2|\Phi_1|^2 dr$ accounting for the contribution of TAE fine scale mode structures. $\chi_0$ can be further simplified, and yields $\chi_0 = \langle\langle\hat{\Lambda}^2\left(\hat{F}_2 - \hat{F}_1^2/\Gamma_S\right)/(\tau\hat{b}_0\sigma_0\Gamma_S)\rangle\rangle$, which is positive definite.

Equation (5) can be considered as the equation describing nonlinear parametric decay of a pump TAE ($\Omega_1$) into TAE ($\Omega_0$) and ion sound mode ($\Omega_S$) daughter waves, which can be solved for the condition of $\Omega_1$ spontaneous decay. Note that $\Omega_S$ could be heavily ion Landau damped, depending on plasma parameter regime such as $\tau$, two parameter regimes with distinct decay mechanisms shall be discussed separately.

For weakly damped $\Omega_S$ due to, e.g., $\tau \gg 1$, both $\Omega_S$ and $\Omega_0$ are normal modes of the system, and the parametric dispersion relation is given as

$$(\gamma + \gamma_S)(\gamma + \gamma_0) = \hat{C}_0|A_1|^2/(\partial_{\omega_S}\varepsilon_{S,R}\partial_{\omega_0}\hat{\varepsilon}_{0,R}), \tag{6}$$

with $\gamma_S$ and $\gamma_0$ being, respectively, the damping rates of $\Omega_S$ and $\Omega_0$, and the subscript "R" denoting real part.

On the other hand, for $\tau \sim O(1)$, $\Omega_S$ is heavily ion Landau damped, and becomes a quasi-mode. One then obtains, from the imaginary part of equation (5),

$$\gamma + \gamma_0 = |A_1|^2\left(\hat{C}_0/|\varepsilon_S|^2 + \chi_0\right)\varepsilon_{S,i}/(\partial_{\omega_0}\hat{\varepsilon}_{0,R}), \tag{7}$$

and the parametric instability $\gamma > 0$ requires $\omega_1 > \omega_0$; i.e., the parametric decay can spontaneously happen only when the pump TAE frequency is higher than that of the sideband, and the parametric decay process leads to, power transfer from higher to lower frequency part of the spectrum, that is, downward spectrum cascading [4].

**TAE spectral transfer and saturation due to nonlinear ion scattering**

Summation over all the background TAEs within the strong interaction region, i.e., counter-propagating and radially overlapping with $\Omega_k$, and the frequency difference $|\omega_k - \omega_{k_1}|$ comparable with ion transit frequency ($|v_i/(qR_0)|$), denoting TAEs with their eigenfrequencies, i.e., $I_k \to I_\omega$, the summation over "$k_1$" can be replaced by integration over "$\omega$", given many background TAEs within the strong interaction range with $\Omega_\omega$ (continuum limit):

$$(\partial_t - 2\gamma_L(\omega))I_\omega = \frac{2}{\partial_\omega\varepsilon_{\omega,R}}\int_{\omega_L}^{\omega_M} d\omega' V(\omega,\omega')I_{\omega'}I_\omega, \tag{8}$$

with $I_\omega \equiv |\nabla A_\omega|^2$, $V(\omega,\omega') \equiv \left(\hat{C}/|\varepsilon_S|^2 + \chi_0\right)\varepsilon_{S,i}/\hat{b}_{\omega'}$, $\omega_M$ being the highest frequency for TAE to be linearly unstable, $\omega_L$ being the lowest frequency for $I_{\omega_L} > 0$, and one has $\omega_M - \omega_L \simeq O(\varepsilon)\omega_T$, comparable with the TAE gap width.

The nonlinear saturation condition can then be obtained from $\partial_t I_\omega = 0$:

$$\gamma_L(\omega) = -(\partial_\omega \varepsilon_{\omega,R})^{-1} \int_{\omega_L}^{\omega_M} d\omega' V(\omega,\omega') I_{\omega'}. \tag{9}$$

Noting that $I_{\omega'}$ varies in $\omega'$ much slower than $V(\omega,\omega')$, and $I_{\omega'} \simeq I_\omega - \omega_S \partial_\omega I_\omega$, we have

$$\gamma_L(\omega) \ = \ (U_0 I_\omega - U_1 \partial_\omega I_\omega)/(2\omega). \tag{10}$$

Here, $U_0 \equiv \int_{\omega-\omega_M}^{\omega-\omega_L} d\omega_S V(\omega_S)$ and $U_1 \equiv \int_{\omega-\omega_M}^{\omega-\omega_L} d\omega_S \omega_S V(\omega_S)$. Noting that, for the ion Compton scattering process to be important, one requires $\omega_M - \omega_L \gg v_{it}/(qR_0)$ and that $V(\omega_S) \propto \varepsilon_{S,i}$ is an odd function of $\omega_S$ varying on the scale of $v_{it}/(qR_0)$, one then obtain

$$I_\omega \ \simeq \ I_M(\omega_M) + \frac{1}{\overline{U_1}} \int_\omega^{\omega_M} \omega \gamma_L(\omega) d\omega, \tag{11}$$

with $I_M(\omega_M) \equiv I_\omega(\omega = \omega_M)$, $\overline{U}_1 \simeq \pi^{3/2} \left(C/|\varepsilon_S|^2 + \chi_0\right) k_{\parallel,S}^2 v_{it}^2/(2\hat{b}_\omega)$. The value of $I_\omega$ at $\omega_M$, $I_M(\omega_M)$, on the other hand, can be determined noting that for $|\omega - \omega_M| \ll |k_{\parallel,S} v_{it}|$ and replacing the lower and upper integral limits of $U_0$ and $U_0$ by, 0 and $\infty$, and one has $U_0(\omega_M) \simeq \overline{U_1}/(k_{\parallel,S} v_{it}$ and $U_1(\omega_M) \simeq \overline{U_1}/2$. $I_M(\omega_M)$ can then be derived from equation (10), noting that $|U_0 I_\omega/(U_1 \partial_\omega I_\omega)| \sim |(\omega_M - \omega_L)/(k_{\parallel,S} v_{it})| \gg 1$, and one has

$$I_\omega = \frac{2k_{\parallel,S} v_{it} \omega_M \gamma_L(\omega_M)}{\overline{U_1}} + \frac{2}{\overline{U_1}} \int_\omega^{\omega_M} \omega \gamma_L(\omega) d\omega. \tag{12}$$

The overall TAE intensity at saturation, can be derived by integrating the intensity over the fluctuation population zone, and we have

$$I_S \equiv \int_{\omega_L}^{\omega_M} I_\omega d\omega \simeq \frac{\overline{\gamma_L}}{\overline{U_1}} \omega_T^3 \left(1 - \frac{\omega_M}{\omega_L}\right)^2. \tag{13}$$

In deriving equation (13), we replaced the TAE linear growth rate $\gamma_L$ with its spectrum averaged value, $\gamma_L(\omega) \simeq \overline{\gamma_L}$, which is validated by the fact that, for burning plasma relevant parameter regimes, a broad TAE spectrum with comparable linear growth rate can be driven unstable. The saturation level of the magnetic fluctuations, can then be derived as

$$|\delta B_r|^2 \sim \frac{\varepsilon^2 \varepsilon_{eff}^2}{2\pi^{3/2}} \frac{\omega_T \overline{\gamma}_L k_r^2}{(\hat{C}/|\varepsilon_S|^2| + \chi_0)\Omega_{ci}^2} \tag{14}$$

with $\varepsilon_{eff} \equiv 1 - \omega_M/\omega_L \sim O(\varepsilon)$ following Ref. [4], and $|k_{\theta,T}/k_{r,T}| \simeq \varepsilon$ for TAEs is assumed. This suggests that, for TAE saturation in the parameter range of practical interest, several processes with comparable scattering cross sections may be equally important [3, 5].

The obtained TAE saturation level and spectrum, can then be applied to derive the ion heating rate from ion nonlinear Landau damping [6] and the EP transport coefficient, which will be reported in a future publication [7].

**References**


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