Practical criteria for the Weibel instability and its saturation

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We consider the Weibel, purely aperiodic instability in a collisionless plasma, relativistic or not, for an important in practice case when the particle distribution functions of all species exhibit mirror symmetry with respect to a certain plane xy and a wave vector of an ordinary wave perturbation is parallel to this plane, i.e., $k_z = 0$ and $f_{0\alpha}(p_x, p_y, p_z) = f_{0\alpha}(p_x, p_y, -p_z)$, where $\mathbf{p} = (p_x, p_y, p_z)$ is the particle momentum. In this case, we obtain a novel analytical criterion for the Weibel instability using its analogy with a long-wavelength soft-mode instability which is well known in the solid state physics. It facilitates an analysis of the Weibel instability and agrees with the results which have been known for the certain particle distributions, including the bi-Maxwellian, power-law, and parallelepiped ones as well as various variants of the so-called waterbag distributions. Also, for a series of special cylindrically-symmetric particle distributions we find the analytical dependence of the Weibel-instability growth rate on the wavenumber of perturbation and show that it agrees well with the criterion presented [1].

The instability is aperiodic and its growth rate increases with wave number k but thereafter vanishes, so that there is a point in the dispersion curve at which the frequency $\omega = 0$ and $k > 0$. If we pass to the limit $\omega \to 0$ in the dispersion equation for an ordinary mode, we find

$$k^2 = -\sum_\alpha \frac{4\pi N_\alpha e_\alpha^2}{c^2} \iiint \frac{f_{0\alpha}}{m_\alpha \gamma_\alpha} \left[1 + \frac{p_z^2}{p_k^2}\right] d^3\mathbf{p}, \tag{1}$$

where $e_\alpha, m_\alpha, \gamma_\alpha, N_\alpha$ are the charge, mass, Lorentz factor and density of particles of the sort α, and $p_k = (\mathbf{pk})/k$. Note, leaving aside regularization of this expression with respect to Cherenkov's singularity of the integrand function at $p_k = 0$ (see below), that the right-hand side of this equation depends on the direction of the wave vector $\mathbf{k}$ but not its modulus. Accordingly, the point with $\omega = 0$, $k \neq 0$ can exist in the dispersion curve at the chosen direction of the wave vector if the right-hand side of Eq. (1) is positive. This equation defines the boundary of the region of wave numbers in which instability is realized, and the condition for its existence takes the form

$$\sum_\alpha N_\alpha e_\alpha^2 \iiint \frac{f_{0\alpha}}{m_\alpha \gamma_\alpha} \left[1 + \frac{p_z^2}{p_k^2}\right] d^3\mathbf{p} < 0. \tag{2}$$

The integrand in Eq. (2) has a singularity requiring a detour in the complex plane. It accounts for the possible negative value of the integral in which the integrand is nowhere negative at real $\mathbf{p}$

values. This singularity does not preclude the application of criterion Eq. (2), because the value of the integral is independent of the way of its detour if the distribution function is smooth.

Equation (2) defines a sufficient condition for the existence of instability if quantity k^2 has a finite value given by Eq. (1) for which $\omega = 0$. Such an instability is realized at least in the vicinity of this k value; generally speaking, it may not be aperiodic and may be accompanied by instability in other wave number ranges to which criterion given by Eq. (2) bears no relation.

The informative value of the sufficient criterion of instability becomes higher and is related to purely aperiodic instability (such as a soft mode with $\mathrm{Re}\,\omega = 0$) when the particle distribution functions f_α exhibit central symmetry $f_\alpha(\mathbf{p}) = f_\alpha(-\mathbf{p})$, which, in particular, guarantees the equality of current density to zero. In this case the dispersion relation takes the form

$$-k^2+\frac{\omega^2}{c^2}-\sum_\alpha \frac{4\pi N_\alpha e_\alpha^2}{c^2}\iiint\left\{\frac{f_{0\alpha}}{m_\alpha\gamma_\alpha}\left[1+\frac{v_{\alpha z}^2(k^2c^2-\omega^2)\left(\omega^2+(\mathbf{k}\mathbf{v}_\alpha)^2\right)}{c^2\left(\omega^2-(\mathbf{k}\mathbf{v}_\alpha)^2\right)^2}\right]-\frac{v_{\alpha z}^2 f_{0\alpha}|_{p_k=0}}{(v_{\alpha k}-i|\sqrt{\omega^2/k^2}|)^2\sqrt{m_\alpha^2+(p^2-p_k^2)/c^2}}\right\}d^3\mathbf{p}=0. \quad (3)$$

The term added to integrand becomes zero upon integration but regularizes the subintegral function, i.e., explicitly neutralizes its singularity. It can be concluded that there is a value of $\omega^2 < 0$ satisfying Eq. (3) at any k smaller than that given by Eq. (1). If 'self-sufficient' condition Eq. (2) is satisfied, instability exists within the entire range of wave numbers from zero to the maximum value given by Eq. (1), wherein it must be aperiodic, i.e., a soft-mode type instability.

To have a rough idea of the physical sense of the above-derived criterion for the Weibel instability Eq. (2), let us consider one sort of particles α responsible for instability. To this end, assume that $k_y = 0$, i.e., direct k along the x-axis and denote $F(p_x, p_y, p_z) = f_{0\alpha}N_\alpha e_\alpha^2/m_\alpha\gamma_\alpha$. Assuming that F as a function of p_x has a maximum at $p_x = 0$, characteristic value F_0, and characteristic width $\tilde{p}_x$, while the expression $(F - F(0, p_y, p_z))/p_x^2$ has a characteristic value $-F_0/\tilde{p}_x^2$ and the same characteristic width $\tilde{p}_x$ allows the instability condition to be approximately rewritten in the form

$$\tilde{p}_x\iint F_0\frac{p_z^2}{\tilde{p}_x^2}dp_y dp_z > \tilde{p}_x\iint F_0 dp_y dp_z. \quad (4)$$

This inequality can be conventionally interpreted as the condition for the root-mean-square components of particle momenta: $\langle p_z^2\rangle > \langle p_x^2\rangle$. In this case, the maximum wave number, i.e., the boundary of the instability region, is estimated as

$$c^2k_{\max}^2 \sim \frac{\omega_p^2}{\tilde{\gamma}}\cdot\left(\frac{\tilde{p}_z^2}{\tilde{p}_x^2}-1\right), \quad (5)$$

where the tilde denotes the characteristic values of the respective quantities.

The most favorable anisotropy corresponds to the elongation of the distribution function across the perturbation wave vector $\mathbf{k} = k\mathbf{x}_0$ and its flattening along it. In general, instability is not aperiodic (Re $\omega \neq 0$) and exists for the entire cone of wave vectors, encompassing the distinguished direction $\mathbf{x}_0$ of plasma anisotropy.

The linear stage of instability development definitely terminates when the newly formed electromagnetic perturbations markedly alter the momentum distribution of the particles responsible for instability and radically mix their trajectories on the field nonuniformity scale, if harmonics are generated within a wide enough wave number range Δk.

To analyze a saturation criterion, for certainty, we consider a single sort of particles. The simplest estimates of the saturation level are obtained on the assumption that the rotation angle of a particle's velocity in the magnetic field $B_{\rm sat}$ being generated becomes close to unity during the time of the order of the reverse growth rate time. Then, the cyclotron frequency in the quasiuniform saturating magnetic field becomes equal to the instability growth rate Γ:

$$\frac{eB_{\rm sat}}{mc\gamma} \sim \Gamma. \tag{6}$$

Substituting the maximum growth rate in the form of $\Gamma \sim (v/c)\omega_p/\sqrt{\gamma}$ and taking into consideration that such a growth rate is realized for the wave numbers $k \sim \omega_p/c\sqrt{\gamma}$ yield

$$\frac{B_{\rm sat}^2/8\pi}{Nmc^2\gamma} \sim \frac{1}{2}\frac{v^2}{c^2}, \qquad \frac{B_{\rm sat}^2/8\pi}{Nmv^2/2} \sim \gamma. \tag{7}$$

This means that the energy of a magnetic field saturating Weibel type instability can be of the same order of magnitude as the particle kinetic energy in both relativistic and non-relativistic plasmas at $\Gamma \sim kv$. If inequalities $\Gamma \gg kv$ or $\Gamma \ll kv$ are satisfied for the wave numbers of harmonics growing with a growth rate of the order of Γ, the saturating (quasiuniform) magnetic field will be weaker and will not reach the 'equipartition' magnitude.

In the difficult-to-realize hypothetical case of $\Gamma \gg kv$, saturation takes place when the change in particle momentum over inverse growth rate time under the effect of the inductive electric field $E_{\rm sat} = \Gamma B_{\rm sat}/kc$ accompanying the appearance of a magnetic field becomes equal to the characteristic particle momentum $mv\gamma$, i.e., $eE_{\rm sat}/\Gamma \sim mv\gamma$. In this case, the saturating magnetic field is thus independent of the growth rate: $B_{\rm sat} \sim mc\gamma kv/e$. This estimate of the onset of the nonlinear stage corresponds to 'magnetization' of plasma particles when their gyroradius becomes equal to the scale of growing large-scale perturbation.

In the case of a small exceedance of the instability threshold (or in the case of weak anisotropy), when $\Gamma \ll kv$, a relatively large-scale magnetic field with the wave numbers $k \ll \omega_p/c\sqrt{\gamma}$ is generated and in the inverse growth rate time most particles are displaced over the distance of many wavelengths of this field (as they move in the magnetic field of variable sign and hence change their velocity direction much more slowly, on average, than in the constant sign field). As a result, the deflection angle for a harmonic in space perturbation is estimated as $(eB/mc\gamma\Gamma)(\Gamma/kv)$; in the case of generation of a large number of random harmonics in a wide wave number range $\Delta k \sim k$, the particles' velocity deflection angle varies in accordance with the diffusive transport and reaches a value of the order of $(eB/mc\gamma\Gamma)\sqrt{\Gamma/kv}$ in the inverse growth rate time. In the last most realistic case, the energy density of the magnetic field at the time of saturation approaches a value around

$$\frac{B_{\text{sat}}^2}{8\pi} \sim \frac{1}{2}N\gamma m v^2 \left(\frac{k^2c^2}{\omega_p^2/\gamma}\right)\left(\frac{\Gamma^3}{k^3v^3}\right), \tag{8}$$

if the saturation condition relies on the equality between the characteristic magnetic field scale $1/k$ and the root-mean-square of the additional particle displacement occurring in the inverse growth rate time due to diffusive velocity fluctuations of the order of ϕv.

Both coefficients in parentheses in Eq. (8) are smaller than unity and define the difference between this estimate and the maximum possible value Eq. (7). Maximum energy density is achieved in magnetic fields with scales around $2\pi/k$ corresponding to the maximum growth rate. These scales are small compared with the gyroradius of free particles. For weak anisotropy, when $\Gamma \ll kv$, the known estimate of energy density in the saturation field, corresponding to the approximate equality of the growth rate Γ to the bounce-oscillation frequency $\sqrt{kveB_{\text{sat}}/\gamma mc}$ of the particles in the vicinity of zero magnetic field regions, is likely to underestimate the true value by a factor of kv/Γ, since, in general, only a small number of particles are subject to bounce-oscillations.

We compare various known estimates of a magnetic field saturating the Weibel instability [1] and, in particular, point to the poorly studied case when this field cannot achieve an equipartition value due to a weak anisotropy of the initial particle distribution, when a relatively large-scale magnetic field is generated. We estimate a number of particles which are subject to bounce-oscillations under these conditions and come to a general criterion of the saturation of the Weibel instability [1]. We show that it is consistent with the analytical results obtained previously for the case of a strong anisotropy as well as with the numerical simulations carried out for the particular examples of a weak anisotropy of particle distribution. Both criteria for the instability and its saturation may be useful for the analysis of typical situations in the space and laboratory collisionless plasmas with anisotropic particle distributions.

References

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