The Spectral Web of the Super-Alfvénic Rotational Instability in accretion disks: An alternative to the MRI paradigm!

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Abstract

The recently developed theory of the *Spectral Web* [1, 2] is a method to compute the full complex spectrum of stationary plasmas together with a connecting structure. This permits to consider the enormous diversity of MHD waves and instabilities of rotating tokamaks and astrophysical plasmas from a single unifying view point.

Presently, the Spectral Web approach is applied to explore the non-axisymmetric rotational instabilities of accretion disks about black holes and neutron stars. These modes are driven by the extremely super-Alfvénic equilibrium flows which are *symmetry breaking* through the dominant Doppler shift $m\Omega$. Here, m is the toroidal mode number and Ω is the angular rotation frequency of the disk. The spectrum of complex modes becomes a very intricate interplay between the real frequencies of the four Doppler-shifted forward and backward Alfvén and slow continua ($\Omega^{\pm}_{\rm A,S} \equiv m\Omega \pm \omega_{\rm A,S}$) and the closely associated complex frequencies of the non-axisymmetric ($m \neq 0$) instabilities. The latter appear as *infinite* sequences 'emitted' from the continua along paths in the complex ω-plane provided by the Spectral Web method. Due to the closeness of the continua, the resulting modes exhibit extreme localization in the radial direction. This is in complete contrast to the standard axisymmetric ($m = 0$) Magneto-Rotational Instabilities (MRIs) [3], where the continua $\omega_{\rm A,S}$ are not Doppler shifted and they do not interact with the MRIs since they are located far away from them in the complex ω-plane. Consequently, the MRIs form a *finite* sequence of unstable eigenvalues, which turn into stable waves when approaching the real axis. Hence, the modes do not have the extreme radial localization that is exhibited by the non-axisymmetric modes. Since the very reason of accretion is generally considered to be the turbulence caused by the Magneto-Rotational Instabilities, it is clear that the non-axisymmetric Super-Alfvénic Rotational Instabilities provide a relevant alternative.

1. The standard Magneto-Rotational Instability

The Magneto-Rotational Instability (MRI) [3] has been proposed by Balbus and Hawley [4] as a mechanism to produce the turbulence, and associated dissipation, needed to explain why, in the obvious absence of ordinary hydrodynamic instabilities, accretion onto compact objects occurs at all. The investigation of this instability most effectively exploits a cylindrical approximation of the disk, where the vertical equilibrium, which produces a rather small variation of the scale height due to the vertical pressure gradient, is neglected. The resulting cylindrical assumption for a thin accretion disk equilibrium is then the approximation $\Phi_{\rm gr} \approx -(GM_\star)/r$ for the gravitational potential of a point mass $M_\star$ in the origin, and the disk is considered to be a thin annular slice of thickness Δz. Such an equilibrium satisfies the differential equation $\rho r(\Omega^2 - GM_\star/r^3) = (p + \frac{1}{2}B^2)' + B_\theta^2/r$, where the angular frequency $\Omega \equiv v_\theta/r$. Since the quantities on the RHS are generally quite small, the rotation of the disk is approximately Keplerian. Such an equilibrium was investigated in the mentioned Ref. [4] for axisymmetric modes in the toroidal direction, $m = 0$, and large values of the wave number k in the vertical direction, $k\Delta z \gg 1$, justifying the cylindrical model.

Exploiting the self-similarity arguments proposed by Spruit *et al.* [5], one obtains the following explicit radial dependencies of the basic variables: $\Omega \sim \rho \sim r^{-3/2}$ and $p \sim B_\theta^2 \sim B_z^2 \sim r^{-5/2}$, whereas the dimensionless amplitudes provide the basic parameters and ordering: $\varepsilon \equiv \sqrt{p_1} \ll 1$, $\beta \equiv 2p_1/B_1^2 \gg 1$, $\mu \equiv B_{\theta 1}/B_{z1} \sim 1$, $\delta \equiv r_2/r_1 - 1 \sim 1$. Here, the indices 1 and 2 refer to the two radii $r = r_1$ and $r = r_2$ of the annular configuration of the disk. The general spectral differential equations for the investigation of the waves and instabilities of such equilibria were obtained by Keppens *et al.* [6]. These equations were implemented in the numerical codes ROC, exploiting the new method of the Spectral Web, described in Ref. [1].

For the MRI, the computed Spectral Web for the equilibrium parameters $\varepsilon = 0.1$, $\beta = 100$, $\mu_1 = 1$, $\delta = 1$ and mode numbers $m = 0$, $k = 70$ is shown in Fig. 13.17 of Ref. [2]. Since our interest here is the non-axisymmetric modes ($m \neq 0$), we will not reproduce that picture here, but just mention the important features for reference. Crucial in all considerations of the MHD spectra of modes *and* instabilities is the analysis of where the continuous spectra are located in the complex ω-plane since those provide the frequencies where the eigenvalues accumulate for large values of the radial mode number, $n \to \infty$. This also relates to the question of local stability. The difference between the axisymmetric modes and the non- axisymmetric modes is enormous in this respect. For the former, there is no Doppler shift so that the continuous spectra are just the static Alfvén continua $\{\pm\omega_{\rm A}(r)\}$ and the static slow continua $\{\pm\omega_{\rm S}(r)\}$. (Since $\beta \gg 1$, the modes are nearly incompressible and those continua approximately coincide.) As follows from the mentioned picture, the axisymmetric modes divide into unstable modes (the MRIs) with a restricted radial mode number range, typically $n \leq 13$, and stable modes with the larger values of n, clustering at the edges of the continua at $-\omega_{\rm A,S}$ and $+\omega_{\rm A,S}$, symmetrically located about the origin and far away from it! Since the origin is approximately where the modes divide into stable and unstable, the MRIs are basically global modes, they conserve the symmetry of the equilibrium, and they do not have a significant propagating wave frequency σ, where the complex eigenvalue parameter $\omega \equiv \sigma + {\rm i}\nu$ is split into σ and the growth rate ν.

2. The Super-Alfvénic Rotational Instability

On the other hand, for the non-axisymmetric modes of the same accretion disk equilibria, the Doppler shift $\Omega_0 = m\Omega \neq 0$ and much larger than the Alfvén and slow frequencies. This yields four Doppler-shifted continua $\{\Omega^{\pm}_{\rm A,S}(r)\}$, where $\Omega^{\pm}_{\rm A,S} \equiv \Omega_0 \pm \omega_{A,S}$, whereas the Alfvén and slow frequencies again virtually coincide due of the large value of β. (Hence, we will omit the subscript S for simplicity of the notation.) Because of the large Doppler shift, the modes are super-Alfvénic (with the multifarious manifestations of 'transsonic' MHD, here just analysed with respect to the linear aspects), giving rise to shocks. For the same reason, both the $+$ and the $-$ continuum modes mainly propagate in the direction of the rotation. Consequently, the resulting spectra of waves and instabilities are dominated by large wave propagation contributions $\sigma \sim \Omega_0 \neq 0$. In particular, the symmetry-breaking of the modes gives rise to sub-spectra of the Super-Alfvénic Rotational Instability (SARI) where both global and local modes clustering at the continua are unstable, in complete contrast with the MRI spectrum.

Fig. 1 shows a Spectral Web of these modes for the same equilibrium and mode numbers as the MRI spectrum mentioned above, except that the width of the disk is chosen smaller ($\delta = 0.1$). This choice is made for reasons of illustration only. Of course, actual disks have a large extension ($\delta \gg 1$), so that their Spectral Webs are even more complex. However, if a cluster spectrum is unstable in a thin radial layer, it certainly will also be unstable in a wider layer. Again, this is in contrast to the MRI spectrum, where the clustering modes are stable.

Recall from [1] that the Spectral Web consists of two sets of curves where the complementary energy is real (*the solution path*, shown in red) or imaginary (*the conjugate path*, shown in blue). The eigenvalues are located at the intersections of those paths, where genuine eigenvalues and false ones alternate. The false ones are easily identified and the genuine ones are indicated by dots in the figure. Because of the large Doppler shift, the two continua $\{\Omega_A^-\}$ and $\{\Omega_A^+\}$ overlap as shown by the inset on top, whereas two branches of the Super-Alfvénic Rotational Instability cluster towards the tips of either one of the continua. Note that the continua themselves are stable (located along the σ-axis), but the clustering modes are unstable ($\nu \neq 0$). The left and right branches of the spectrum are labelled 'inner' and 'outer' to distinguish the kind of localization of the corresponding eigenfunctions (shown in Fig. 2 for two representative cases). The 'inner' modes cluster towards the tip of the forward Alfvén continuum $\{\Omega_A^+\}$ and the 'outer' modes cluster towards the tip of the backward continuum $\{\Omega_A^-\}$. Note that they both localize about the average Doppler frequency, but for the 'inner' modes their range is 'cut off' on the outside by the singularity of the backward Alfvén singularity for $\sigma = \Omega_A^-(r_{\rm cutoff})$, whereas the forward Alfvén singularity for $\sigma = \Omega_A^+(1.0^-)$ is situated just left from the interval (see Fig. 1(a)). For the 'outer' modes the roles of Ω_A^- and Ω_A^+ are just reversed and the modes are 'cut off' on the inside (see Fig. 1(b)). Hence, effectively, the two singularities act as a kind of boundary conditions for the modes: beyond them the amplitudes of the perturbations are virtually negligible!

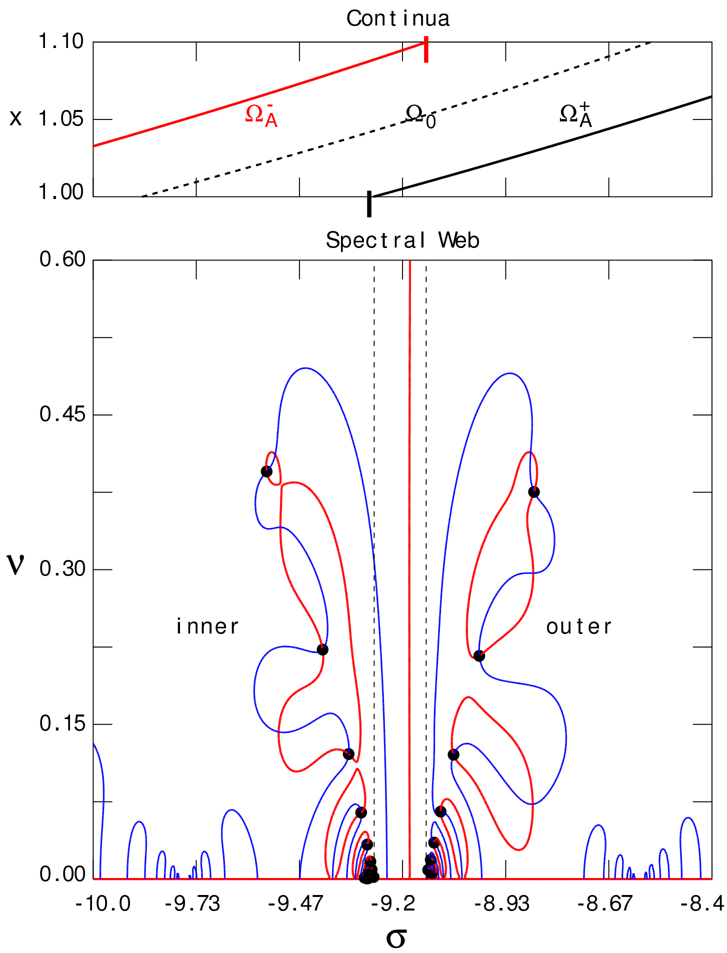

Figure 1: *Spectral Web of the two branches of the Super-Alfvénic Rotational Instability for $m = -10$, $k = 70$. The inset on top shows the radial profiles of the continua.*

This 'trapping' of the modes by the continuum singularities is one of the surprising features of the new Super-Alfvénic Rotational Instability which distinguishes it from the standard Magneto-Rotational Instability. Clearly, the trapping just depends on the radial distribution of the Doppler-shifted continua $\{\Omega_A^-\}$ and $\{\Omega_A^+\}$. Since these are robust features of the magnetohydrodynamical description of accretion disks, the new modes should be at least as effective in producing turbulence and dissipation as the standard ones.

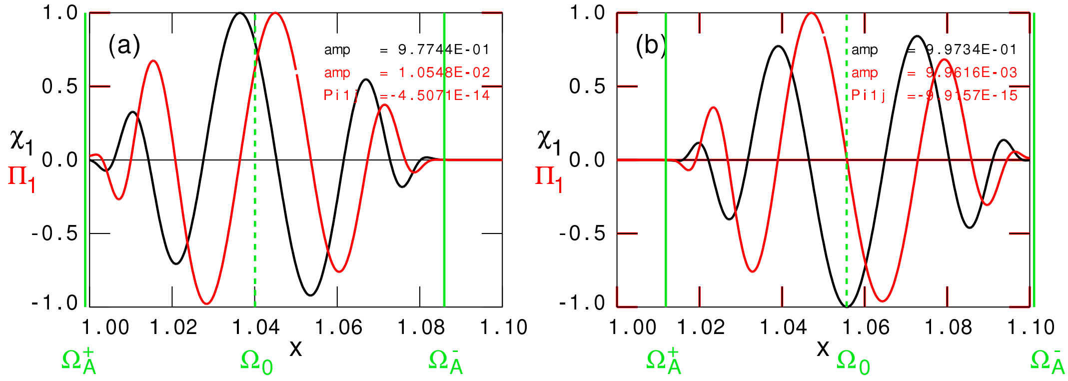

Figure 2: *Two eigenfunctions of the Super-Alfvénic Rotational Instability corresponding to Fig. 1: (a) 'Inner' mode (approaching the tip of the continuum* $\{\Omega_A^+\}$*),* $\sigma = -9.3080$*,* $\nu = 0.06518$*; (b) 'Outer' mode (approaching the tip of the continuum* $\{\Omega_A^-\}$*),* $\sigma = -9.1031$*,* $\nu = 0.06368$.

3. Conclusions

- *The Spectral Web is a powerful tool for the investigation of all instabilities of plasma equilibria with large rotational velocities, as in accretion disk about compact objects.*
- *The axisymmetric Magneto-Rotational Instabilities are assumed to produce the turbulence, and associated dissipation, needed for accretion onto massive central objects.*
- *If so, the non-axisymmetric Super-Alfvénic Instabilities enormously enlarge the possible dynamics of accretion disks and, hence, for turbulence and dissipation.*

References

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