Gyrokinetic theory of toroidal Alfvén eigenmodes nonlinear saturation

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Shear Alfvén wave (SAW) instabilities such as toroidal Alfvén modes (TAE) [1] are expected to play important roles in magnetic confinement fusion devices as energetic particles (EPs) contribute significantly to the total power density [2]. TAEs can be driven unstable by EPs, and in turn, induce EP transport and degrade overall plasma confinement. In burning plasmas of future reactors with EP characteristic orbit width much smaller than the minor radius, most unstable TAEs are characterized by toroidal mode number $n \sim O(10)$. As a result, many TAEs coexist and nonlinear mode couplings, e.g. ion induced scattering [3], as a potential channel for TAE nonlinear saturation, are important for the qualitative and quantitative understanding of EP confinement in future tokamaks.

The condition for TAE saturation via ion induced scattering can be satisfied in burning plasmas, since 1. there are many ($\sim O(n^2 q)$) TAEs with radially overlapping mode structures, 2. TAE frequency is independent of the toroidal mode number and can be roughly estimated by $\omega \sim V_A/(2qR_0)$ with the frequency deviation being $\sim O(\varepsilon\omega)$, and 3. TAEs are characterized by $|k_\parallel| \simeq 1/(2qR_0)$ in the inertial (radially fast varying) layer where nonlinear coupling dominates. In the scattering process, a pump TAE may decay into a counter-propagating TAE sideband and an ion sound quasi-mode (ISW) with much lower frequency and $k_\parallel \simeq 1/(qR_0)$. The nonlinear evolution depends sensitively on the thermal ion thermal to magnetic pressure ratio β_i. In the low β_i limit with $\beta_i \ll \varepsilon^2$, ion transit frequency is much smaller than TAE frequency gap width, and the pump TAE decays into a TAE lower sideband and ISW daughter wave; while in the high β_i limit, the sideband is a propagating lower kinetic TAE (LKTAE) in the continuum. The low-β_i limit is investigated in great detail in Ref. [6], and, thus, in this work we focus on the high-β_i limit. Note that, ISW frequency is higher for $T_e \gg T_i$ and, in this "high-β limit", resonant decay into weakly ion Landau damped ISW is preferred. Thus, higher order terms associated with Ω_S can be neglected in the analysis. The present theory considers the short wavelength kinetic regime, with $\omega/\Omega_{ci} \ll k_\perp^2 \rho_i^2$ for burning plasmas of fusion interest, and the perpendicular couplings due to Reynolds and Maxwell stresses dominate the parallel ponderomotive forced induced by the $\mathbf{b}\cdot\delta\mathbf{J}\times\delta\mathbf{B}$ nonlinearity. This will lead to a lower TAE saturation level and, consequently, lower EP transport than the prediction based on MHD limit [3].

The analysis presented here and in Ref. [6], derived for TAE, can be applied to other toroidal Alfvén modes (TAMs), i.e., SAW instabilities in the TAE frequency range, strongly affected by toroidal effects, including TAE, kinetic TAEs (KTAE), as well as energetic particle modes (EPMs). As an example, in ITER plasma with $V_\alpha \sim 2V_A$, upper KTAE (UKTAE) could be preferentially excited, and decay into a TAE due to ion induced scattering, providing a potential nonlinear saturation channel for UKTAE.

Theoretical model

To investigate the nonlinear decay of a pump TAE, $\Omega_1 = (\omega_1, \mathbf{k}_1)$, decaying into a LKTAE, $\Omega_0 = (\omega_0, \mathbf{k}_0)$, and a low n ISW, $\Omega_S = (\omega_S, \mathbf{k}_S)$, the standard nonlinear perturbation theory is adopt. The scalar potential $\delta\phi$ and parallel vector potential $\delta A_\parallel$ are used as the field variables, and one has, e.g., $\delta\phi = \delta\phi_0 + \delta\phi_1 + \delta\phi_S$, with the subscripts 0, 1 and S denoting LKTAE, pump TAE and ISW, respectively. Furthermore, $\delta\psi \equiv \omega\delta A_\parallel/(ck_\parallel)$ is taken as an alternative field variable, and with $\delta\psi = \delta\phi$ in the ideal MHD limit. Without loss of generality, $\Omega_0 = \Omega_1 + \Omega_S$ is adopted as the frequency/wavenumber matching condition. For effective spectral transfer by nonlinear ion Landau damping, we have $|\omega_S| \sim O(v_{it}/qR_0)$, i.e., the ISW frequency is comparable to thermal ion transit frequency. Therefore, Ω_0 and Ω_1 are counter-propagating (along the magnetic field line), with $\omega_0 \simeq \omega_1$ and $k_{\parallel,0} \simeq -k_{\parallel,1}$.

The governing equations describing the nonlinear interactions among Ω_0, Ω_1 and Ω_S, can then be derived from quasi-neutrality condition and nonlinear gyrokinetic vorticity equation [4], while the nonadiabatic particle responses can be derived from nonlinear gyrokinetic equation [5].

Nonlinear parametric instability

The nonlinear generation of ISW due to Ω_0 and Ω_1 beating can be derived to be

$$\varepsilon_S\delta\phi_S = i(\hat{\Lambda}/\omega_0)\sigma_0\sigma_1\delta\phi_0\delta\phi_{1^*}, \tag{1}$$

where $\hat{\Lambda} \equiv (c/B_0)\hat{\mathbf{b}}\cdot\mathbf{k}_0\times\mathbf{k}_{1^*}$, $\varepsilon_S \equiv 1+\tau+\tau\Gamma_S\xi_S Z(\xi_S)$ is the linear dispersion function of Ω_S, with $\tau \equiv T_e/T_i$, $\Gamma_S \equiv \langle J_S^2 F_0/n_0\rangle$, $\xi_S \equiv \omega_S/(k_{\parallel,S}v_{it})$ and $Z(\xi_S)$ is the plasma dispersion function. Furthermore, $\beta_1 \equiv \sigma_0\sigma_1 + \tau\hat{F}_1\left(1+\xi_S Z(\xi_S)\right)$, with $\hat{F}_1 \equiv \langle J_0 J_1 J_S F_0/n_0\rangle$, $\sigma_k \equiv 1+\tau-\tau\Gamma_k$.

The LKTAE polarization can be derived by substituting linear/nonlinear particle responses of Ω_0 into the quasi-neutrality condition, and one has $\delta\psi_0 = \sigma_0\delta\phi_0 + D_0\delta\phi_1\delta\phi_S$, with $D_0 \equiv i\hat{\Lambda}\tau\hat{F}_1\left[1+\xi_S Z(\xi_S)\right]/\omega_0$. The nonlinear eigenmode equation of Ω_0 can be derived from vorticity equation as

$$\hat{\varepsilon}_0\delta\phi_0 \;=\; i\frac{\omega_0\hat{\Lambda}}{b_0}(\Gamma_S-\Gamma_1)\delta\phi_1\delta\phi_S. \tag{2}$$

Here, $\varepsilon_0 \equiv \varepsilon_T(\Omega_0)$ is the WKB linear dispersion relation of Ω_0, with $\varepsilon_T \equiv k_{\parallel,T}^2 V_A^2 \sigma_T - (1-\Gamma_T)\omega_T^2/\hat{b}_T$. The LKTAE eigenmode dispersion relation can then be derived noting the $V_A^2 \propto 1 - 2(r/R_0+\Delta')\cos\theta$ dependence on poloidal angle θ with Δ' being Shafranov shift, and properly accounting for the kinetic effects.

Substituting equation (1) into (2), multiplying both sides of the obtained equation with Φ_0^*, noting that ε_S varies much slower than $|\Phi_0|^2$ and $|\Phi_1|^2$ in radial direction, and integrating over the radial domain, one then has

$$\varepsilon_S \hat{\varepsilon}_0 = -\langle\langle \sigma_0\sigma_1\hat{\Lambda}^2(\Gamma_S-\Gamma_1)/b_0\rangle\rangle |A_1|^2,, \tag{3}$$

in which $\hat{\varepsilon}_0$ is the linear TAE eigenmode dispersion relation, defined as $\hat{\varepsilon}_0 = \int |\Phi_0|^2 \varepsilon_0 dr$, and $\langle\langle\cdots\rangle\rangle \equiv \int \Phi_0^2\Phi_1^2(\cdots)dr$. Equation (3) is thus, the local nonlinear parametric dispersion relation, describing a pump TAE (Ω_1) decay into LKTAE (Ω_0) and ISW (Ω_S) daughter waves, which can be solved for the condition of Ω_1 spontaneous decay.

The sideband Ω_0 is a radially propagating LKTAE in the lower continuum, and $\hat{\varepsilon}_0$ can be written as [7]

$$\hat{\varepsilon}_0 = -\frac{\pi k_\theta^2 \rho_i^2 \omega_A^2}{2^{2\hat{\xi}+1} b_0 \Gamma^2(\hat{\xi}+1/2)}\left[\frac{2\sqrt{2}\Gamma(\hat{\xi}+1/2)}{\hat{\alpha}\Gamma(\hat{\xi})} + \delta W_f\right].$$

Here, $\Gamma(\hat{\xi})$ and $\Gamma(\hat{\xi}+1/2)$ are Euler gamma functions, $\hat{\xi} \equiv 1/4 - \Gamma_+\Gamma_-/(4\sqrt{\Gamma_-\hat{s}^2\hat{\rho}_K^2})$, $\Gamma_\pm \equiv \omega^2/\omega_A^2(1\pm\varepsilon_0) - 1/4$, $\omega_A^2 \equiv V_A^2/(q^2R_0^2)$, $\hat{\alpha}^2 = 1/(2\sqrt{\Gamma_-\hat{s}^2\hat{\rho}_K^2})$, $\hat{s} \equiv r\partial_r q/q$ is the magnetic shear, δW_f playing the role of a potential energy, and $\hat{\rho}_K^2 \equiv (k_\theta^2\rho_i^2/2)\,[3/4+\tau(1-i\delta_e)]$ denotes kinetic effects associated with finite ion Larmor radii and electron Landau damping, including trapped electron collisional damping.

Equation (3) can be solved following the standard procedure of resonant decay instabilities, and yields

$$(\gamma+\gamma_0)(\gamma+\gamma_S) = \left(\langle\langle\sigma_0\sigma_1\hat{\Lambda}^2(\Gamma_S-\Gamma_1)/b_0\rangle\rangle|A_1|^2\right) / \left(\partial_{\omega_0}\bar{\hat{\mathscr{E}}}_{0,R}\partial_{\omega_S}\bar{\mathscr{E}}_{S,R}\right). \tag{4}$$

The selection rules for the decay mode number is determined by the short radial scale averaging in equation (4).

Nonlinear saturation and EP transport

The resulting TAE saturation level, can be derived following the analysis of Ref. [7], where TAE decay into GAM and LKTAE is analyzed. For the simplicity of discussion, consistent with the approach above, we assume Ω_S is weakly ion Landau damped. The equation for the feedback of Ω_0 and Ω_S to the unstable pump TAE Ω_1, is derived as

$$\mathscr{E}_1\delta\phi_1 = i\frac{\omega_1}{b_1}\hat{\Lambda}(\Gamma_S-\Gamma_0)\delta\phi_0\delta\phi_{S^*}. \tag{5}$$

The three-wave nonlinear dynamic equations can then be derived as

$$(\partial_t + \gamma_S)A_S = \hat{\alpha}_S A_0 A_{1^*}, \quad (6)$$

$$(\partial_t + \gamma_0)A_0 = \hat{\alpha}_0 A_1 A_S, \quad (7)$$

$$(\partial_t - \gamma_1)A_1 = \hat{\alpha}_1 A_0 A_{S^*}, \quad (8)$$

with γ_1 being the linear growth rate of the linearly unstable pump TAE due to, e.g., EP drive, $\hat{\alpha}_S \equiv \int dr \Phi_0 \Phi_{1^*} \hat{\Lambda} \sigma_0 \sigma_1 / (\omega_0 \partial_{\omega_s} \bar{\mathscr{E}}_{S,R})$, $\hat{\alpha}_0 \equiv \omega_0 \int dr |\Phi_0|^2 |\Phi_1|^2 dr \hat{\Lambda}(\Gamma_S - \Gamma_1)/(b_0 \partial_{\omega_0} \bar{\hat{\mathscr{E}}}_{0,R})$ and $\hat{\alpha}_1 \equiv \omega_1 \int dr |\Phi_0|^2 |\Phi_1|^2 dr \hat{\Lambda}(\Gamma_S - \Gamma_0)/(b_1 \partial_{\omega_1} \bar{\mathscr{E}}_{1,R})$. The above coupled equations, describing the nonlinear evolution of the driven-dissipative system, may exhibit rich dynamics such as limit-cycle behaviors, period-doubling and route to chaos as possible indication of the existence of strange attractors. In this work, focusing on TAE nonlinear saturation and related transport, the TAE saturation level can then be estimated from the fixed point solution as $|A_1|^2 = \gamma_0 \gamma_S / (\hat{\alpha}_S \hat{\alpha}_0)$. Note that, the present analysis, assuming ISW being weakly ion Landau damped, can be readily generalized to ISW heavily ion Landau damped parameter regime, by taking $\gamma_S \simeq v_{it}/(qR_0)$. The corresponding magnetic fluctuation amplitude, can then be estimated in the $b \leq 1$ limit, and one obtains

$$\left| \frac{\delta B_r}{B_0} \right|^2 \simeq \frac{2\gamma_0 \gamma_S}{\omega_0 \omega_S} \frac{\varepsilon^2 k^2_{\parallel,0}}{k^2_{\theta,1}} \sim 6.5 * 10^{10} A_m \frac{\gamma_0 \gamma_S}{\omega_0 \omega_S} \varepsilon^2 R_0^{-2} B_0^{-2} T_E. \quad (9)$$

The TAE saturation level in the high-β limit, can be estimated as $|\delta B_r / B_0| \sim 10^{-4}$ for typical ITER-like parameters, assuming $\gamma_0/\omega_0 \sim 10^{-2}$ and $\gamma_S/\omega_S \sim 1$. The corresponding EP diffusion rate due to resonance overlapping can be derived using quasi-linear transport theory, and one has the local diffusion coefficient of well circulating EPs being $D \sim 1 m^2/s$. For obtaining the particle flux, the global theory is needed, with the effects of radial envelope properly accounted for; which will be reported in a future publication

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