Electromagnetic modelling of the Reversed Field Pinch configuration

Roberto Cavazzana

Consorzio RFX (CNR, ENEA, INFN, Università di Padova, Acciaierie Venete SpA)
Padova, Italy

1. Introduction - The problem of describing the evolution of the *Reversed Field Pinch* (RFP) configuration in connection with the magnetic coils and their power supply circuits has been tackled in the past in a number of papers, the notable ones being ref.s [1, 2]. Nevertheless even if these start from clever and smart approaches, some mistakes are repeatedly present in the derivations, giving rise to a number of more or less subtle problems in the model implementation and inconsistencies in the final results. In this short paper a more rigorous derivation is obtained, starting from fundamental electromagnetic laws. The RFP equilibrium of choice, can be then inserted in the evolution equations only at a later stage of the derivation. This approach leads to clean equations in explicit formulation, which can be used for immediate practical purposes like the calculation of the dissipative loop voltage, or to be inserted directly in a SPICE circuit simulator to predict the macroscopic behaviour of the RFP experimental discharges.

2. Internal, external and MHD equilibrium magnetic fields - At the essential the electrical elements which define a plasma experiment consists of a bunch of distributed currents inside the plasma producing the field $\mathbf{B}^*_{\mathbf{int}}$ and some arrays of field coils, producing the field $\mathbf{B}_{\mathbf{ext}}$. Obviously to describe a meaningful plasma the internal currents have to produce a field which once combined with the external one, satisfies the MHD equilibrium equations. Because of the linearity of the Ampère's law, the total equilibrium MHD field $\mathbf{B}_{\mathbf{MHD}}$ comes up to be the sum of internal and externally generated fields:

$$\mathbf{B}_{\mathbf{MHD}} = \mathbf{B}^*_{\mathbf{int}} + \mathbf{B}_{\mathbf{ext}}. \tag{1}$$

The basic method presented here is quite general, although the focus is on the simplified case of a cylindrical axi-symmetric pinch plasma contained in a uniform toroidal coil, matching the plasma surface. Using the standard procedure of refs.[1, 2], consider a plasma with radius a and periodicity $2\pi R$ contained in a volume V, with right handed coordinate system (r,θ,z) -a notation resembling the usual pseudo-toroidal coordinates commonly used in pinch experiments-; the plasma becomes separated from external systems at the boundary surface ∂V. Given this representation, the Poynting's theorem allows to identify the relevant electrical quantities linked to the average magnetic and electric field at the boundary ∂V: $\frac{\partial}{\partial t}\int_V u\, dV + \oint_{\partial V} \mathbf{S}\, d\mathbf{A} = -\int_V \mathbf{J}\cdot\mathbf{E}\, dV$. Here $u = \frac{B^2_{MHD}}{2\mu_0}$ is the magnetic energy density and $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ the Poynting vector, which in the case of the cylindrical pinch is written as:

$$\mathbf{S} = (E_\theta B_z - E_z B_\theta)\,\hat{\mathbf{r}} + (E_z B_r - E_r B_z)\,\hat{\boldsymbol{\theta}} + (E_r B_\theta - E_\theta B_r)\,\hat{\mathbf{z}}$$

Assuming negligible the integral contributions from spatial and radial fluctuations at the boundary, its integral can be written as:

$$\oint_{\partial V} \mathbf{S}\, d\mathbf{A} = \frac{1}{\mu_0}(2\pi a \cdot 2\pi R)(E_\theta(a)B_z(a) - E_z(a)B_\theta(a)) \tag{2}$$

Formula 2 outlines the four average integrals of the boundary electromagnetic field which in turn define the four fundamental electrical variables measurable on pinch experiments:

plasma current $$I_{pla} = \frac{2\pi a}{\mu_0} B_\theta(a) \tag{3a}$$

toroidal loop voltage $$V_{loop} = 2\pi R E_z(a) \tag{3b}$$

toroidal winding current $$I_{tor} = \frac{2\pi R}{\mu_0} B_z(a) \tag{3c}$$

poloidal loop voltage $$V_\theta = 2\pi a E_\theta(a) \tag{3d}$$

3. RFP two port network definition - Given the above quantities (eq. 3) the generic pinch (and of course the RFP) can be seen as a generic two-port network (Fig. 1). It is worth noticing that I_{tor} corresponds to the total single turn poloidal currents flowing *outside* the plasma, while I_{pla} results as the integral of the *internal* toroidal components of the current flowing *inside* the plasma. The naming chosen here follows the engineers' conventional jargon: in experimental environment I_{tor} corresponds to the sum of the poloidal currents flowing into the *toroidal windings* and passive structures, setting the average toroidal field applied at the edge of the plasma $B_t(a)$. This simple observation shows that the reversal of the RFP comes from the direction of the externally applied current I_{tor}, in contrast to the common jargon spread in the RFP community (namely *the plasma reverses the field*): the reversal condition is the equilibrium response of the plasma to an imposed value of the toroidal field at its edge and not an intrinsic property of the plasma alone. Another remark pertains the fact that I_{tor} and I_{pla} themselves do not carry any information about the internal poloidal current distribution of the plasma J_θ, or equivalently anything about the internal toroidal flux. This information is embedded in the plasma inductance and in the mutual coupling coefficient, as will be outlined later on.

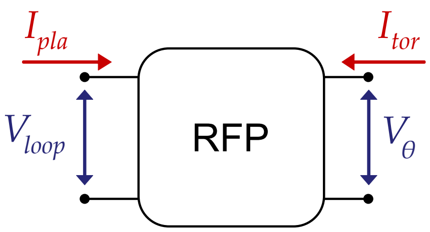

Figure 1: *Two-port network representation of the RFP and its relevant electrical quantities.*

4. Explicit equation of the two-port network - The two-port network equation constraining the evolution of MHD equilibrium can be inferred by carefully analyzing the structure of the magnetic energy, momentarily leaving out the dissipation. From the electrical perspective the pinch plasma acts as a non-linear transformer-like device, with some mutual coupling between the two ports. Recalling the expression for the magnetic energy stored into a generic system of K mutual coupled inductors $U_m(\bar{I}) = \frac{1}{2}\sum_{i,j=1}^{K} I_i L_{ij} I_j$, the equation for the RFP two-port network reads:

$$U_m = \tfrac{1}{2} L_p I_{pla}^2 + M I_{tor} I_{pla} + \tfrac{1}{2} L_t I_{tor}^2 \tag{4}$$

As with any other electrical device of this type, the plasma system has three separate contributors to its magnetic energy: the internal magnetic field generated by plasma currents, the externally applied field (uniform for the cylindrical RFP) and the coupling between these two.

From the MHD configuration point of view the *total* magnetic energy U_m is obviously:

$$U_m = U_{MHD} = \frac{1}{2\mu_0}\int_V B^2_{MHD}\, dV\,. \tag{5}$$

The internal self-generated field produced by the currents flowing inside the plasma $\mathbf{B}^*$ can be simply found recalling eq. 1, by subtracting from the total MHD equilibrium field the externally generated one: $\mathbf{B}^* = \mathbf{B_{MHD}} - \mathbf{B_{ext}}$. Now eq. 5 can be restated to recover the structure of eq. 4:

$$U_{MHD} = \frac{1}{2\mu_0}\int_V \left[B^{*2} + 2\left(\mathbf{B}^{*}\cdot\mathbf{B_{ext}}\right) + B^2_{ext}\right] dV. \tag{6}$$

In the case of the cylindrical pinch the poloidal field is entirely generated by the currents inside the plasma, so $B^*_\theta(r) = B_\theta(r)$. The internal generated toroidal field is instead obtained by subtracting the externally applied one: $B^*_z(r) = B_z(r) - B_{z\,ext}(r)$; again because of the cylindrical symmetry the field $B_{z\,ext}(r) = B_z(a)$, in this case constant along r.

Now the inductances L_p, M and L_t can be obtained from the proper contribution of each source. The plasma inductance L_p is readily defined from the first term of eq. 6: $\frac{1}{2}L_p I^2_{pla} = \frac{1}{2\mu_0}\int_V \left(B^2_\theta + B^{*2}_z\right) dV$: once the fields in this expression are normalized to the edge poloidal field $\mathbf{b}^*(r) = \frac{\mathbf{B}^*(r)}{B_\theta(a)}$ and the I_{pla} is eliminated by recalling eq.3a, the plasma inductance becomes then correctly defined solely on the geometrical properties of the configuration:

$$\boxed{L_p = \mu_0 R\,\frac{1}{a^2}\int_0^a \left(b^2_\theta(r) + b^{*2}_z(r)\right) r dr} \tag{7}$$

By comparison refs [1, 2] erroneously derive the plasma inductance directly from the total energy of the equilibrium field U_{MHD}.

The second term of eq. 6 corresponding to the coupling inductance M can be obtained without explicit calculation of its integral, invoking the usual quantities used to identify the RFP equilibria $F = \frac{B_z(a)}{\langle B_z\rangle}$ and $\Theta = \frac{B_\theta(a)}{\langle B_z\rangle}$, which are defined as the ratio between the magnetic field components at plasma boundary and the average axial field $\langle B_z\rangle = \frac{\Phi_z}{\pi a^2} = \frac{2}{a^2}\int_0^a B_z r dr$. The self generated toroidal flux of the plasma $\Phi^*_z = \Phi_z - \pi a^2 B_z(a)$ [3] can be written in terms of these quantities; since $B_z(a) = \frac{F}{\Theta}B_p(a) = \frac{F}{\Theta}\frac{\mu_0 I_{pla}}{2\pi a}$, M becomes: $\boxed{M = \frac{\Phi^*_z}{I_{pla}} = \frac{\mu_0 a}{2}\left(\frac{1-F}{\Theta}\right)}$ (8). Here M turns out to be the parameter which summarizes the RFP self organization properties: it links the *internal self-generated* flux by the internal poloidal currents J_θ at a given plasma current I_{pla}.

The last term L_t is simply the one of an ideal single turn inductor with the same dimensions of the plasma, i.e. the "toroidal" vacuum inductance, independent on the presence of the plasma: $\boxed{L_t = \frac{\mu_0 \pi a^2}{2\pi R}}$ (9) . Finally the law governing the inductive voltages at the two-port network can be laid down from Faraday-Lenz's law, taking into account both voltage components induced by the variation of the currents *and* the change of magnetic configuration:

$$\begin{bmatrix} V_{loop} \\ V_{pol} \end{bmatrix} = \begin{bmatrix} L_p & M \\ M & L_t \end{bmatrix}\begin{bmatrix} \dot{I}_{pla} \\ \dot{I}_{tor} \end{bmatrix} + \begin{bmatrix} \dot{L}_p & \dot{M} \\ \dot{M} & 0 \end{bmatrix}\begin{bmatrix} I_{pla} \\ I_{tor} \end{bmatrix} \tag{10}$$

This formulation can be used by coupling this equation to any sort of MHD model or experimental measurements describing the evolution of plasma equilibrium, and by closing the boundary condition with the appropriate external circuit elements. While this explicit formula can be directly inserted in electrical network solvers such as SPICE, the solution proposed in [2] results in a implicit formulation, difficult to implement and prone to numerical instability [4]. Moreover in ref.s [1, 2] the voltage evolution is incorrectly derived from the Poynting theorem, obtaining an incomplete equation in the form $V = L\dot{I} + \frac{1}{2}\dot{L}I$, which omits the energy associated to the mechanical work made by the magnetic field [5]. In the case of a plasma the neglected work component corresponds to the energy associated to the shifting of magnetic tension $(\mathbf{B}\cdot\nabla)\mathbf{B}$.

5. Resistive loop voltage - The resistive loop voltage for RFP experiments can be retrieved using the two port network evolution eq. 10 along with the Poynting theorem:

$$\frac{\partial}{\partial t}\int_V u\,dV + \oint_{\partial V}\mathbf{S}\,d\mathbf{A} = -\int_V \mathbf{J}\cdot\mathbf{E}\,dV \tag{11}$$

The ohmic dissipation term is recovered once the magnetic part of the Poynting theorem is balanced. Since I_{tor} does not pass through the plasma and cannot contribute to the dissipation, in the right term of eq.11 only the terms containing I_{pla} survive: V_{res} is thus defined writing: $V_{res}I_{pla} = \int_V \mathbf{J}\cdot\mathbf{E}\,dV$ (12). Putting togheter eq.s 10, 11 and 12, the dissipative loop voltage equation for V_{res} finally reads: $\boxed{V_{res} = V_{loop} - \left[L_p\dot{I}_{pla} + \dot{L}_p I_{pla}\right] - \left[M\dot{I}_{tor} + \dot{M}I_{tor}\right]}$ (13). The inductances L_p (7) and M (8) along with their time derivatives are to be obtained from the time evolution of MHD equilibria reconstructed from experimental measurements.

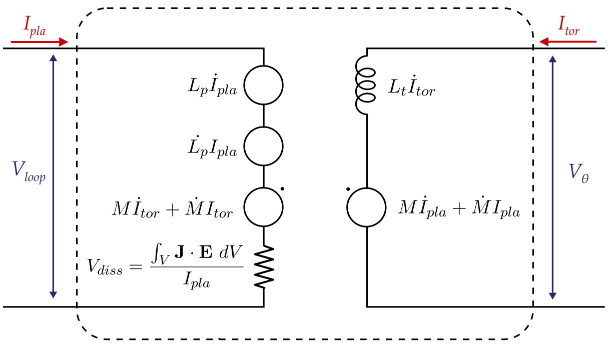

Figure 2: *Complete electrical scheme of the two port network RFP.*

Eq. 13 is the correct one to quantify the power balance during transient phases, such as the discrete magnetic reconnections observed in the RFP. Moreover it outlines the internal structure of the two-port network in its detailed electrical equivalent scheme (Fig. 2). It is worth to recall that the Poynting theorem as written in eq. 11 already includes in the $\mathbf{J}\cdot\mathbf{E}$ term the dissipation associated to the viscous damping of the fluid components of the plasma flow.

References

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