Hamiltonian Formulation of the Non-perturbative Guiding Centre Equation

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Abstract

Guiding Centre (GC) theory aims to reduce the dynamics of a charged particle in given electric and magnetic fields, by exploiting the huge disparity in the time-scales involved in the problem. Here we consider a geometric approach in which the GC is defined as the reference frame in which a particle moves in a closed orbit in phase-space. This introduces a new set of coordinates, and thus a new set of equations of motion, for the problem. In this work a hamiltonian structure is proposed for the new equations of motions, in the relativistic regime.

We consider the problem of determining the motion of a charged particle in a given external electromagnetic field. The particle is isolated and its retroaction on the fields is assumed negligible. The motion of the charged particle in some (laboratory) reference frame is described in a phase space $\mathscr{M}$ of coordinates $\{t, x, v\}$, and follows some vector field C. In the Poincaré-Cartan formulation of mechenics, the vector field is determined by $d\varpi C = 0$, where[1]

$$\varpi = \overline{p}dx - \varepsilon dt$$

is the Poincaré-Cartan one-form. Here p and ε are respectively the canonical momentum and the energy of the particle.

Qualitatively, the motion of the charge is a helix around the magnetic field lines. This helicoidal motion is usually much faster than any other time-scale involved. GC theory has been studied since the sixties as a method to "average out" the gyration, and to get reduced equations. In the most common approach, this reduction is accomplished perturbatively, eventually with the aid of a gauge trasform [3], [4]. In [1], [2] a new intrinsic approach proposed. It is still based on a gauge trasform on phase space, but the GC coordinates are defined geometrically, while the GC velocity is the solution of a particular PDE. This equation can be solved exactly,

[1] We denote by an overbar the standard Euclidean structure: if v is a (column) vector, $\overline{v}$ is the trasposed of it, i.e. the corresponding row vector. We call it "covector". Contraction is inteded when a covector is followed by a vector. A vector followed by a covector is a matrix.

in any field geometry; for instance, in [1] a solution is found for a typical tokamak field. If the equation is solved perturbatively, it reproduces the known results. In this work we study how the hamiltonian structure of the charged particle dynamics changes under the change of variables to GC coordinates. We derive a new set of equations and recover the PDE for the GC velocity as a defining equation.

The Non-Perturbative GC Theory

We define the *Guiding Centre* as the reference frame in which the charged particle is seen moving in a closed orbit in phase space. The GC time T and position X are the time and position coordinates of this reference frame in the laboratory frame. Moreover we define the *gyrophase* γ as the angle parametrizing the orbit in the GC reference frame. The GC transform $\mathscr{T}$ is the map conjugating the original phase space coordinates $\{t,x,v\}$ to the new set $\{T,X,\gamma,\mu,\varepsilon\}$. This map is chosen so that the Poincaré-Cartan form transforms as follows,

$$\varpi_{GC} \equiv -\mathscr{E}\,dT + \overline{P}\,dX - \tfrac{m}{e}\,\mu\,d\gamma \tag{1}$$

Here $\mathscr{E}$ and P are functions (fields) of $\{T,X,\varepsilon,\mu\}$, defined by

$$P(T,X,\varepsilon,\mu) = U(T,X,\varepsilon,\mu) + \tfrac{e}{m}A(T,X), \quad \mathscr{E}(T,X,\varepsilon,\mu) = U^0(T,X,\varepsilon,\mu) + \tfrac{e}{m}\Phi(T,X)$$

where U and U^0 are also functions, such that $\dot{X} = U$ and the overdot means derivation with respect to proper time. The *magnetic moment* μ is defined by equation (1) itself as the momentum associated to the gyrophase. Then we have[2]

$$\begin{aligned} d\varpi_{GC} =& d\varepsilon\,\partial_\varepsilon\mathscr{E}\wedge dT + d\mu\,\partial_\mu\mathscr{E}\wedge dT - \tfrac{e}{m}d\overline{X}E_c\wedge dT - d\varepsilon\,\partial_\varepsilon\overline{P}\wedge dX + \\ & - d\mu\,\partial_\mu\overline{P}\wedge dX - \tfrac{e}{m}d\overline{X}\,[[B_c]]\,dX + \tfrac{m}{e}d\mu\wedge d\gamma \end{aligned} \tag{2}$$

where we defined the "canonical magnetic field" $(e/m)B_c \stackrel{\text{def}}{=} [[\nabla]]P$ and the "canonical electric field" $(e/m)E_c \stackrel{\text{def}}{=} -\nabla\mathscr{E} - \partial_T P$. We look for a vector $C = U^0\partial_T + \overline{U}\nabla + \dot{\gamma}\partial_\gamma + \dot{\mu}\partial_\mu + \dot{\varepsilon}\partial_\varepsilon$ in the null space of this two-form:

$$\begin{aligned} d\varpi_{GC}C =& d\varepsilon\,\partial_\varepsilon\mathscr{E}\,U^0 - dT\,\partial_\varepsilon\mathscr{E}\,\dot{\varepsilon} + d\mu\,\partial_\mu\mathscr{E}\,U^0 - dT\,\partial_\mu\mathscr{E}\,\dot{\mu} - d\overline{X}E_c U^0 + \tfrac{e}{m}dT\,\overline{U}E_c + \\ & + \tfrac{e}{m}d\overline{X}\,[[B_c]]\,U - d\varepsilon\,\partial_\varepsilon\overline{P}U + d\overline{X}\,\partial_\varepsilon P\,\dot{\varepsilon} - d\mu\,\partial_\mu\overline{P}U + d\overline{X}\,\partial_\mu P\,\dot{\mu} + \tfrac{m}{e}d\mu\,\dot{\gamma} - \tfrac{m}{e}d\gamma\,\dot{\mu} \stackrel{!}{=} 0 \end{aligned}$$

[2]Still about notation: if v is a vector, then $[[v]]$ is the skew-symmetric matrix corresponding to the vector product with it. Coherently, $[[\nabla]]$ is the curl operator.

The expression above is equivalent to the five equations

$$\dot{\mu} = 0 \tag{3a}$$

$$-U^0 E_c + [[B_c]]U + \tfrac{m}{e}\dot{\varepsilon}\partial_\varepsilon U = 0 \tag{3b}$$

$$-\overline{E_c}U + \tfrac{m}{e}\dot{\varepsilon}\partial_\varepsilon U^0 = 0 \tag{3c}$$

$$\tfrac{m}{e}\dot{\gamma} = -U^0\,\partial_\mu U^0 + \overline{U}\partial_\mu U \tag{3d}$$

$$\partial_\varepsilon (U^0)^2 = \partial_\varepsilon(\overline{U}U) \tag{3e}$$

where we have deleted a term in (3b) as a consequence of (3a), and we have used $\partial_\mu P = \partial_\mu U$, $\partial_\varepsilon P = \partial_\varepsilon U$, $\partial_\mu \mathscr{E} = \partial_\mu U^0$, $\partial_\varepsilon \mathscr{E} = \partial_\varepsilon U^0$. There is a redundancy in the system above: for instance, equation (3e) can be derived by multiplying (3b) on the left by $\overline{U}$ and (3c) by U^0.

Hamiltonian Formulation

To give a hamiltonian formulation to the problem means to choose one coordinate to parametrize the others; for instance here we choose T. Then one performs a splitting of the Lagrange form as $d\varpi = \mathscr{H}\wedge dT + \sigma$ where $\mathscr{H}$ is the "Hamiltonian form". The splitting is meaningfull as long as σ is closed and injective; if so, σ is called the "simplectic form", and it admits an "inverse" π such that $\sigma Y \pi = Y$ for any vector field Y (not involving the T coordinate). We call π "Poisson two-vector". The dynamics is determined by a vector $C = \partial_T + Y$ so that

$$d\varpi C = \mathscr{H} + \sigma Y = 0 \quad \Longrightarrow \quad Y = -\mathscr{H}\pi$$

From the Lagrange form (2) we get the symplectic form,

$$\sigma = -d\varepsilon\,\partial_\varepsilon \overline{P}\wedge dX - d\mu\,\partial_\mu\overline{P}\wedge dX + \tfrac{e}{m}d\overline{X}[[B_c]]dX + \tfrac{m}{e}d\mu\wedge d\gamma \tag{4}$$

and the hamiltonian form,

$$\mathscr{H} = d\varepsilon\,\partial_\varepsilon\mathscr{E} + d\mu\,\partial_\mu\mathscr{E} - \tfrac{e}{m}d\overline{X}E_c \tag{5}$$

Determining the Poisson structure

In the GC problem, π has to solve the following equations

$$(\sigma\nabla)\pi = \nabla \qquad (\sigma\partial_\gamma)\pi = \partial_\gamma \qquad (\sigma\partial_\varepsilon)\pi = \partial_\varepsilon \qquad (\sigma\partial_\mu)\pi = \partial_\mu \tag{6}$$

We consider the following ansatz:

$$\pi = \xi\partial_\varepsilon\wedge\partial_\gamma + \overline{\nabla}\mathscr{A}\nabla + \partial_\varepsilon\wedge\overline{z}\nabla + \tfrac{e}{m}\partial_\mu\wedge\partial_\gamma + \partial_\gamma\wedge\overline{s}\nabla$$

with vector fields z and s, a scalar function ξ and an skew-symmetric matrix $\mathscr{A}$ all to be determined. In principle we may also include a term coupling ∂_μ and ∇, and another one coupling ∂_μ

and ∂_ε, but they turn out to be always zero, so we omitted them from the beginning. We define $b \stackrel{\text{def}}{=} B_c/|B_c|$. A possible solution of equations (6) is

$$\mathscr{A} = \frac{m}{e|B_c|}[[b]], \qquad z = -\frac{V_b}{\partial_\varepsilon \mathscr{E}} b, \qquad \partial_\varepsilon P = \frac{\partial_\varepsilon \mathscr{E}}{V_b} b, \quad s = 0, \qquad \partial_\mu U = 0 \tag{7}$$

where we introduced $dX/dT \equiv V = U/U^0$ and $V_b = \overline{V} b$. We end up with

$$\pi = \frac{e}{m} \partial_\mu \wedge \partial_\gamma + \frac{m}{e|B_c|} \overline{\nabla} [[b]] \nabla - \partial_\varepsilon \wedge \overline{b} \nabla \tag{8}$$

which is equivalent to the Lie bracket

$$\{F, G\} = \tfrac{e}{m}\big(\partial_\mu F \cdot \partial_\gamma G - \partial_\gamma F \cdot \partial_\mu G\big) + \frac{m}{e|B_c|} \overline{\nabla F} [[b]] \nabla G + \big(\overline{b} \nabla F \cdot \partial_\varepsilon G - \partial_\varepsilon F \cdot \overline{b} \nabla G\big)$$

Hamilton's equations

With Hamiltonian (5) and Poisson two-vector (8) one gets

$$-\mathscr{H}\pi = V_b \overline{b} \nabla - \frac{e}{m}\left(\partial_\mu \mathscr{E}\right) \partial_\gamma + \frac{\overline{E_c}[[B_c]]}{|B_c|^2} \nabla + \frac{e}{m} V_b \frac{\overline{b} E_c}{\partial_\varepsilon \mathscr{E}} \partial_\varepsilon \tag{9}$$

Then, the vector field above is equivalent to the equations of motion

$$\tfrac{d}{dT} X = V_b b + [[E_c]] B_c / |B_c|^2 \tag{10a}$$

$$\tfrac{d}{dT} \gamma = -\tfrac{e}{m} \partial_\mu \mathscr{E} \tag{10b}$$

$$\tfrac{d}{dT} \varepsilon = \tfrac{e}{m} V_b \overline{b} E_c / \partial_\varepsilon \mathscr{E} \tag{10c}$$

$$\tfrac{d}{dT} \mu = 0 \tag{10d}$$

Now we have to check the compatibility of the system (10) with (3). This check is based on the equality $\frac{d}{ds} = \frac{dT}{ds}\frac{d}{dT} = U^0 \frac{d}{dT}$. So, for instance, equations (3d) and (10b) are compatible (in fact, they are equivalent) in virtue of the latter equation of (7). Then equation (3b) can be rewritten as

$$E_c - [[B_c]] V - \frac{d\varepsilon}{dT} \partial_\varepsilon U = 0 \implies V = V_b b + \frac{[[E_c]] B_c}{|B_c|^2} \tag{11}$$

where a term proportional to $[[B_c]] \partial_\varepsilon U$ was erased as a consequence of the third equation of (7). What remains is then compatible with (10a). Thirdly, equation (3c) can be rewritten as $d\varepsilon/dT = \overline{E_c} V / (\partial_\varepsilon U^0)$ and is compatible with equation (10c) if $\overline{E_c} V = V_b \overline{E_c} b$ which in turn is a consequence of equation (11).

As a final note, we stress that equation (11) is the PDE for the GC velocity that we mentioned in the introduction.

References

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