Analytic equilibrium of elongated plasmas bounded by a magnetic separatrix and the problem of resistive axisymmetric X-point modes

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Abstract. *Theoretical and experimental considerations suggest that axisymmetric perturbations that are resonant at the X-point(s) of a magnetic divertor separatrix may play a role for the understanding of ELMs in tokamaks and their active control via so-called vertical kicks. The first step in the development of an analytic model for resistive axisymmetric X-point modes is presented, i.e., finding an adequate and relatively simple analytic MHD equilibrium for a plasma column with noncircular cross section bounded by a magnetic separatrix.*

The stability of the X-point region of tokamak plasmas with divertor configurations takes on great importance in the realization of sustainable H-mode regimes. The toroidal magnetic field line going through the X-point is resonant to axisymmetric MHD perturbations[1]. Thus, when resistivity is accounted for, localized current sheets can be driven unstable, leading to a change of the X-point topological structure. Such process has been studied extensively in the context of astrophysical plasmas[2]. Therefore, one may suspect that in a tokamak the magnetic X-point region may be strongly influenced by axisymmetric MHD perturbations. From an experimental viewpoint, observations correlate Type-I giant ELMs with n=0 axisymmetric perturbations. A puzzling experimental fact was the rather large shift of the strike points on divertor target plates observed during giant ELMs in JET experiments[3]. It was suggested[1] that this large shift could be explained by the inferred[3] relatively large currents flowing from the magnetic X-point to the target plates near both strike points. Another case in point is the question of the *vertical kicks* experiments for active ELM control[4]. It may be argued that vertical kicks could lead to a change of the X-point topology. Yet, in the context of tokamak plasmas, it appears that the theory of resonant, resistive axisymmetric X-point modes has not been developed so far. In this article, we are concerned with the first step in the development of an analytic model for these modes, i.e., finding an adequate and relatively simple analytic MHD equilibrium for a plasma column with noncircular cross section bounded by a magnetic separatrix[5]. An early example is Gajewski's solution[6], which we extend by considering finite external currents located at finite distance from the magnetic X-points. Another relevant equilibrium solution was proposed recently by Xu and Fitzpatrick[7].

In this article, the cylindrical plasma column extends along the ignorable coordinate *z*, which mimics the toroidal angle φ of a tokamak configuration. The current density J_z is taken to be uniform in space up to a nearly (but not exactly) elliptical cross section with major axis *b* and minor axis a. The external currents are modeled by two equal, parallel current filaments also directed along the *z*-axis and placed symmetrically at distance *l* from the plasma center, as shown in Fig. 1. The X-point coordinates are $\pm l_X$, where $b < l_X < l$. We introduce a small expansion parameter,

$$\rho_0 = \frac{b^2-a^2}{l^2} \tag{1}$$

The external currents are found to satisfy the condition

$$\frac{I_{ext}}{I_p} = \frac{l^4}{(a+b)^2(a^2+b^2)}\rho_0 = \frac{b-a}{b+a}\frac{l^2}{(a^2+b^2)} \tag{2}$$

where I_{ext} is the external current carried by either one of the two filaments and $I_p = \pi abJ_z$ is the plasma current. Gajewski's equilibrium[6] is recovered in the limit $\rho_0 \to 0$ and $l \to \infty$. In this limit, the surface bounding the plasma current becomes exactly elliptical.

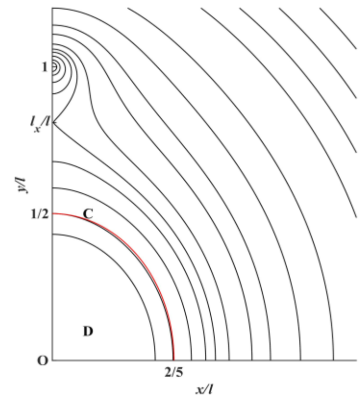

Figure 1. Magnetic flux surfaces. Because of symmetry, only the quadrant with positive x and y is shown. Parameter values are a/l=2/5, b/l=1/2, yielding ρ_0=0.09. The red line corresponds to the ellipse with semi-axis a and b.

The magnetic field is $\boldsymbol{B} = B_z\boldsymbol{e_z} + \boldsymbol{e_z} \times \nabla\psi$, where $\boldsymbol{e_z}$ is a unit vector. The current density is $\mu_0\boldsymbol{J} = \nabla \times \boldsymbol{B} = \nabla B_z \times \boldsymbol{e_z} + \nabla^2\psi\boldsymbol{e_z}$. The ideal MHD equilibrium condition is $\nabla p = \boldsymbol{J} \times \boldsymbol{B}$. From these equations, $J_z = J_z(\psi)$ and therefore $\nabla^2\psi = \mu_0 J_z(\psi)$ is a nonlinear equation for ψ to be solved subject to appropriate boundary conditions. Furthermore, $p = p(\psi)$ and $B_z = B_z(\psi)$. Details of the solution procedure are given in Ref. [5]. Here, we outline the main steps. In solving the equilibrium problem, we note that this contains three functions of ψ, i.e., $J_z(\psi)$, $p = p(\psi)$ and $B_z = B_z(\psi)$, of which two can be chosen arbitrarily and the third is derived consistently with this choice. We choose $J_z(\psi)$ = constant inside the domain *D* of the Oxy plane centered at the origin, $x = y = 0$, bounded by the curve $C(x,y) = 0$ (see Fig. 1), which corresponds to an ellipse to leading order in ρ_0.

The choice $J_z = J_0 = const$ inside D converts the equilibrium problem into a linear, inhomogeneous partial differential equation. Thus, the equations to be solved are

$$\nabla^2\psi = \mu_0 J_0 \qquad \text{inside domain } D \tag{3}$$

$$\nabla^2\psi = \mu_0 I_{ext} \cdot [\delta(x, y-l) + \delta(x, y+l)] \qquad \text{outside domain } D, \tag{4}$$

with boundary conditions $\psi = \psi_0 = const$ on curve $C(x,y) = 0$, $\{\psi\} = 0$, $\{\partial\psi/\partial n\} = 0$, where the angular brackets $\{\phi\}$ denote the jump of the generic quantity ϕ across the boundary of *D* and $\boldsymbol{n}$ is the outer normal. Also, $\psi \sim \ln r$ for $r = (x^2+y^2)^{1/2} \to \infty$, which ensures that the equilibrium magnetic field goes to zero as r^{-1} for $r \to \infty$. The difference between our approach and Gajewski's procedure[6] can be summarized as follows: (i) Curve *C* is exactly an ellipse in [6], while it differs from an ellipse by terms of order ρ_0 in our work; (ii) $\nabla^2\psi = 0$ outside domain *D* in [6], while Eq. (4) is used here; (iii) the condition $\psi \sim \ln r$ for $r \to \infty$ is enforced here.

It is convenient to introduce elliptical coordinates (μ, ϑ), related to Cartesian coordinates by $x = A\ \sinh\eta\ \sin\vartheta$; $y = A\cosh\eta\ \cos\vartheta$. We set $a = A\ \sinh\eta_0$; $b = A\cosh\eta_0$, then $A^2 = b^2 - a^2$ and $\eta = \eta_0$ defines the boundary $C(x,y) = 0$ of the plasma current to zeroth order in the parameter ρ_0. This boundary is an ellipse with semi-axes *a* and *b*. Using the superposition principle, the solution for the flux function can be written as $\psi(x,y) = \psi_P(x,y) + \psi_{ext}(x,y)$, where $\psi_P \propto I_P$ is the magnetic flux generated by the plasma current flowing inside domain *D* and $\psi_{ext} \propto I_{ext}$ is the magnetic flux generated by the two external current filaments. The boundary of *D* must be a magnetic flux surface of constant ψ. However, $\psi_P(x,y)$ and $\psi_{ext}(x,y)$ are not constant over the boundary, only their sum will be. This requirement imposes the relationship between I_P and I_{ext} anticipated in Eq. (2).

After straightforward algebra, the details of which are given in [5], we find the solution to leading order in ρ_0:

$$\psi(x,y) = \psi_0 \cdot \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) \quad \text{inside domain } D;$$

$$\psi(\eta,\vartheta) = \frac{\mu_0}{4\pi} I_P \left\{1 + 2(\eta - \eta_0) + e^{-2\eta}\cos(2\vartheta) + \frac{l^4}{(a+b)^2(a^2+b^2)}\rho_0 \ln\left\{1 + \rho_0[1 + \cosh(2\eta)\cos(2\vartheta)] + \frac{1}{4}\rho_0^2[\cosh(2\eta) + \cos(2\vartheta)]^2\right\}\right\} \quad \text{outside domain } D.$$

An example of this solution is given in Fig. 1, where surfaces of constant ψ are drawn for $\rho_0 = 0.09$. We can see that the boundary of domain *D* is closely approximated by the ellipse $\eta = \eta_0$ (the red curve in the figure). To find the magnetic separatrix, we first solve for the coordinates of the X-points, where $\nabla\psi = 0$, and subsequently look for the surface of constant flux passing through such points. In Gajewski's limit[6] (subscript "G"), one finds $\eta_{XG} = 2\eta_0$, or equivalently $l_{XG} = (a^2 + b^2)/(b^2 - a^2)^{1/2}$, where l_X is the distance of the X-point from the origin of the Oxy plane. Carrying out the calculation to first order in ρ_0, we find $\eta_X = \eta_{XG} - \rho_0\eta_{X1}(\delta)$, where $\delta = a/b$ and $\eta_{X1}(\delta) = (1+\delta^2)^3/[(1-\delta^2)(1+\delta)]^2$, or equivalently, $l_X = \{1 - [2\rho_0\eta_{X1}(\delta)\delta]/(1+\delta^2)\}l_{XG}$. Figure 2 shows a graph of l_X/l_{XG}, obtained numerically from the full solution in Eq. (23), as function of ρ_0 for fixed $\delta = 0.8$. The curves stops at $\rho_0 = \rho_*$, which corresponds to the limit where $l_X = b$. The equilibrium solution ceases to be valid for of $\rho_0 > \rho_*$.

In conclusion, we have extended Gajewski's solution[6] for the equilibrium of a plasma column bounded by a magnetic separatrix to the case where the external currents are located symmetrically at finite distance from the boundary of the plasma current density and the latter is distributed uniformly over a domain D bounded by a nearly elliptical magnetic flux surface[5]. Three main results are found: (i) The analysis relies on a small expansion parameter, ρ_0, defined in Eq. (1); to leading order in ρ_0 the boundary of domain D can be approximated by an ellipse. (ii) Equilibrium requires that the external currents, I_{ext}, be related to the plasma current according to the criterion in Eq. (2). (iii) The geometric structure and topology of the magnetic flux surfaces depend on two parameters only: a/l and $\delta = a/b$, where b and a are the major and minor semi-axes of the elliptical boundary, respectively, and $\pm l$ are the coordinates of the two external current filaments on the y-axis. Gajewski's equilibrium is recovered in the limit $l \to \infty$.

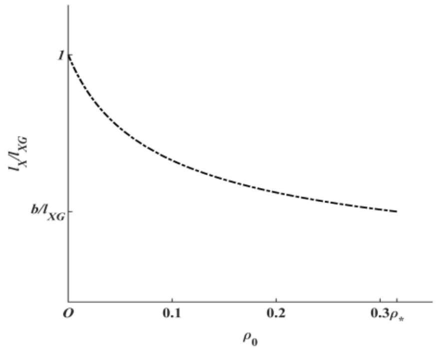

Figure 2. X-point coordinate, l_X, normalized to Gajewski's limiting value l_{XG}, as function of ρ_0 and for fixed δ = 0.8 and ρ_* = 0.32.

This equilibrium is expected to be unstable to ideal MHD *vertical* displacements of the plasma column. Just like in the case of tokamak plasmas with elongated cross section, we can also expect that modulating in time the external currents can stabilize the vertical instability. This would mimic the passive feedback stabilization scenario of a tokamak plasma, where time-dependent image currents are induced on the conducting wall containing the plasma. Of more interest will be to study the case of a resistive plasma extending to the magnetic separatrix. A vertical plasma displacement would be resonant at the magnetic X-points, giving rise to the possibility that current sheets centered at the X-points be driven unstable. In the equivalent tokamak scenario, this type of perturbation is what we refer to as resistive axisymmetric X-point modes[1,5]. The treatment of these modes will be the subject of a future investigation.

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