SPH simulation of cylindrical and toroidal MHD systems

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The present contribution is motivated by the lack of tools to study the stability properties of plasma configurations that include magnetic islands and stochastic regions. These equilibrium solutions do not assume the existence of nested closed magnetic surfaces and can not be easily tackled analytically, instead codes like SIESTA [1] construct them numerically. As an example, Wendelstein 7-X depends on our capabilities to keep the X-points of its islands and the divertors aligned, hence the pressing need to understand the stability properties of such complex configurations.

Smooth Particle Hydrodynamics, or SPH for short, is a Lagrangian numerical method designed to solve the equations of hydrodynamics commonly used to simulate galaxy dynamics, and solar plasmas by the astrophysical community. The method was first introduced in the 70's by Gingold and Monaghan [2] together with Lucy [3]. In SPH all fields are "carried by the particles" and are evaluated via interpolation formulas, in contrast to what is done in other methods such as particle-in-cell (PIC) codes. This interpolation procedure allows us to discretise the spatial derivatives on a co-moving frame and to obtain evolution equations for the particle's position, velocity, mass density, internal energy and magnetic field.

$$\rho_a = \sum_b m_b W_{ab}(H_a) \tag{1}$$

$$\frac{dv_a^i}{dt} = -\sum_b m_b \left(\frac{\mathbb{S}_a}{\Omega_a \rho_a^2} F_{ab}(H_a) + \frac{\mathbb{S}_b}{\Omega_b \rho_b^2} F_{ab}(H_b) \right) \cdot \mathbf{r}_{ab} \tag{2}$$

$$\frac{dB_a^i}{dt} = -\sum_b m_b \left(\mathbf{B}_a (\mathbf{v}_{ab} \cdot \mathbf{r}_{ab}) - \mathbf{v}_{ab} (\mathbf{B}_a \cdot \mathbf{r}_{ab}) \right) \frac{F_{ab}(H_a)}{\rho_a \Omega_a} \tag{3}$$

$$\frac{du_a}{dt} = -\frac{p_a}{\rho_a} \sum_b m_b (\mathbf{v}_{ab} \cdot \mathbf{r}_{ab}) \frac{F_{ab}(H_a)}{\rho_a \Omega_a} \tag{4}$$

Evolution equations of the system (eqs.1, 2, 3 and 4)[1] are constructed by identifying the Lagrangian of the system, and minimising the corresponding action functional [4]. The resulting equation of motion poseses conservation properties linked to the invariants of the Lagrangian function which are responsible for most of the success that SPH has had lately.

[1]Here, W is gaussian-like interpolating kernel, The notation $\mathbf{r}_{ab} = \mathbf{r}_a - \mathbf{r}_b$ and similarly for the velocity and the magnetic field. $W_{ab} = W(|\mathbf{r}_{ab}|)$ and $\nabla W_{ab} = -\mathbf{r}_{ab} F_{ab}$. The factor Ω is included to account the spatially varying H-field. Finally we have $\mathbb{S} = -(p + B^2/2\mu_0)\mathbb{I} + \mathbf{BB}/\mu_0$

Using SPH to simulate the non-linear temporal evolution of three-dimensional MHD systems with fusion-relevant geometries requires us to overcome the following challenges: Prescribe and enforce appropriate boundary conditions on curved conducting walls and replicate arbitrary initial conditions with high-fidelity. Solutions to these challenges have been recently developed (See [5], [6] and [7]) and have allowed us to tackle cylindrical and toroidal scenarios.

For cylindrical geometries, we consider a Theta-pinch and a Zeta-pinch configuration as listed in [8]. We initialise the particles using the ALARIC algorithm to place them in an homogeneous initial density profile (see fig. 1).

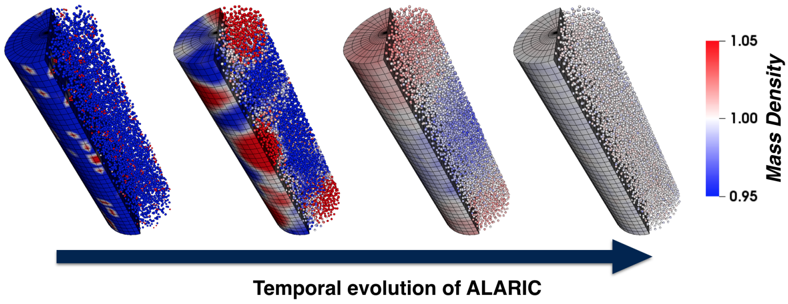

Figure 1: *Cylindrical pinch Initial Condition*

Their stability properties can be evaluated analytically leading to: Unconditional stability for the theta pinch and instability for the zeta pinch for poloidal wave numbers $m \in \{0,1\}$ and aggravated by large azimuthal wave numbers $k \mapsto \infty$. These results are illustrated in fig.2 where the final snapshot (at $t = 7t_A$) of the Theta-pinch radial profiles (left) and the constant pressure iso-surfaces of the plasma column (right) are shown. Here the profiles have remain unchanged. In contrast, fig.3 shows a Zeta-pinch after a similar period of time and we can see how the radial profiles, initially a thin line, become noisy due to the radial projection of the kinked plasma column (rightmost part of fig.3).

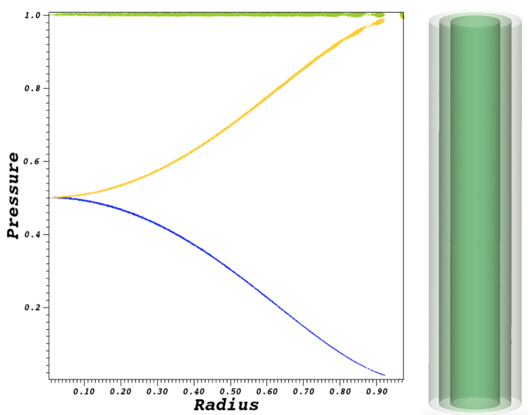

Figure 2: *Theta Pinch final snapshot*

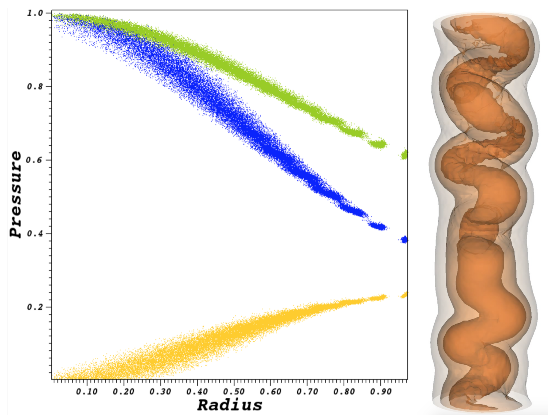

Figure 3: *Zeta Pinch final snapshot*

For toroidal geometries we have performed a set of simulations with aspect ratio $R_0/a = 3$ and circular cross section where the initial condition (Soloviev equilibrium profiles) have been constructed on top a flat density profile obtained with ALARIC (see fig.4)

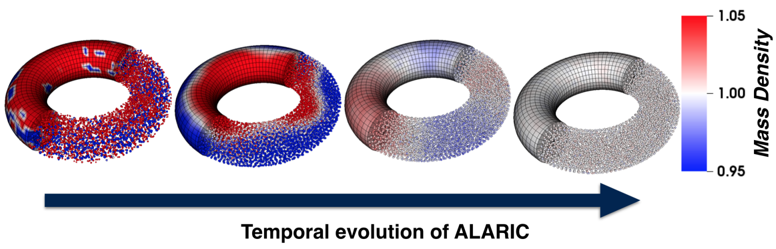

Figure 4: *High-beta tokamak Initial Condition*

We initialized the system with the following particular equilibrium solutions to the Grad-Shafranov equation:

$$B_R = -\frac{1}{R}\frac{\partial \psi}{\partial Z} \quad B_Z = +\frac{1}{R}\frac{\partial \psi}{\partial R} \quad B_\Psi = \frac{F(\psi)}{R} \quad \text{and} \quad \psi = \frac{A}{4}(r^2 - a^2) + \frac{C}{8}(r^2 - a^2) r \cos(\theta)$$

This particular solution[2] is illustrated in fig. 5 and is only an approximate solution to the Grad-Shafranov equation. That is, at $t = 0$ the forces are not completely balanced (See 6).

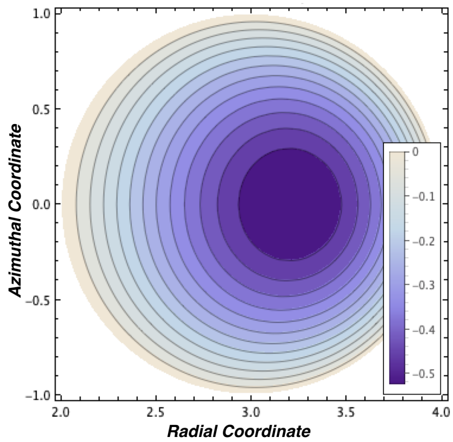

Figure 5: *Equilibrium solution ψ_0*

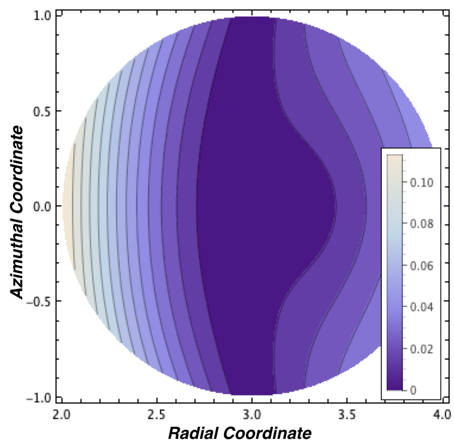

Figure 6: *Force balance at $t = 0$*

The the temporal evolution of this initial condition does not present any macroscopic changes during the simulation period. In fact, by looking at fig.7 and fig.8 no significant changes in the topology of the magnetic configuration can be seen.

However, a closer look at the time trace of the kinetic energy reveals a small oscillation that seems to damp and converge to a true equilibrium. The precise signal in fig. 9 seems to be

[2]Here, the following considerations have been used:

$$F^2(\psi) = R_0^2 B_0^2 \left[1 - \frac{2\mu_0 p(\psi)}{B_0^2} + 2\frac{\tilde{B}(\psi)}{B_0}\right] \qquad R_0^2 B_0 \frac{d\tilde{B}}{\psi_0} = -A \qquad 2\mu_0 R_0 \frac{dp}{d\psi_0} = -C$$

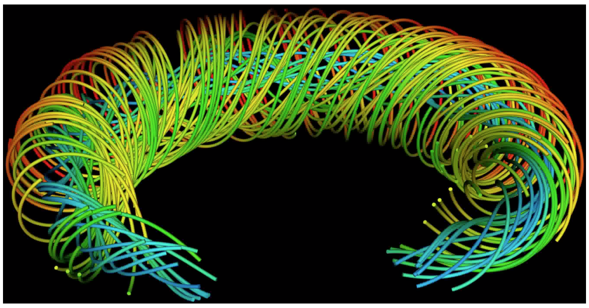

Figure 7: *Magnetic streamlines at* $t = 0$

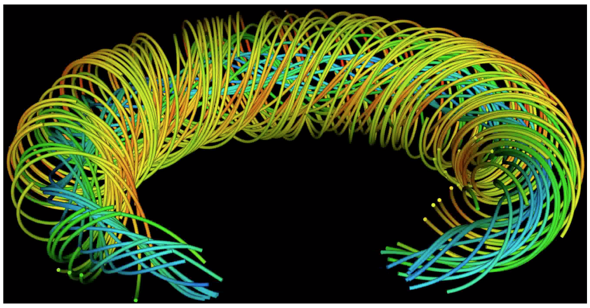

Figure 8: *Magnetic streamlines at* $t = 10\tau_A$

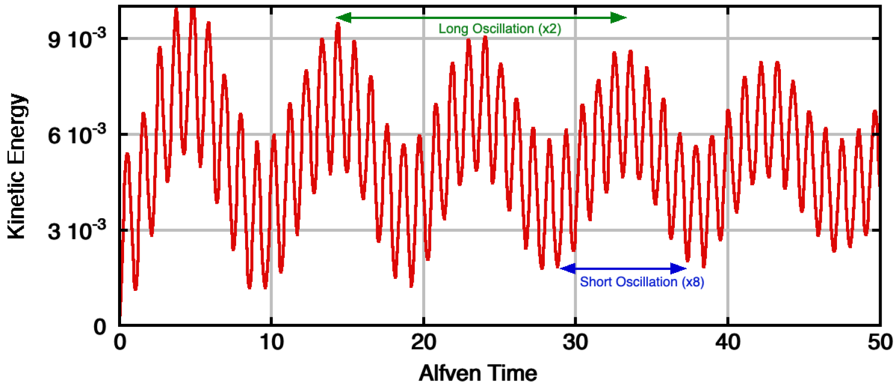

Figure 9: *Kinetic Energy time trace for Soloviev's solution*

composed of two main components: A short oscillation with an approximate period of τ_A and a longer oscillation ($9\tau_A$). The amplitude of the oscillation presents a damped/convergent behaviour that hints at the proximity of a stable configuration around which the systems oscillates.

The results obtained here show that SPH can be used to simulate non-linear dynamics of MHD systems in fusion-relevant scenarios including toroidal geometries with circular cross-section. Furthermore, the particular simulation of the Soloviev equilibrium seems to suggest a stable configurations that would oscillate towards a true equilibrium. In this respect, the prospect of using SPH as a numerical tool to find true equilibria is suggested.

References

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