One dimensional kinetic model of an inverted sheath in a bounded plasma system

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A one-dimensional kinetic model of an inverted sheath [1] is presented. The model is based on a bounded plasma system model, introduced by Schwager and Birdsall [2]. A one-dimensional bounded plasma system is considered. The system is bounded by two very large planar electrodes perpendicular to the *x* axis. The left hand electrode is called collector and the right hand electrode is called the source. The collector is located at $x = 0$ and the source is located at $x = L$. The source is at zero potential, $\Phi_S = \Phi(x = L) = 0,$ which is taken as the reference. The collector is electrically floating with respect to the source. The source injects electrons (plasma electrons) and singly charged positive ions into the system, both with half-Maxwellian velocity distributions and different temperatures. Collector absorbs all the particles that reach it, but it also emits electrons (emitted electrons) also with half-Maxwellian velocity distribution and their own temperature. It is assumed that potential decreases monotonically from the collector to the source. Based on this assumption and the assumption that the energy of the particles is conserved cut-off Maxwellian distribution functions of all 3 particle species can be ‘’derived’’ in the following way. An electron that has left the source with negligibly small velocity, will have at the position *x*, where the potential is $\Phi(x)$ the velocity $\sqrt{2e_0\Phi(x)/m_e}$ in the direction of negative *x* axis towards the collector. So the distribution of the source electrons at the position *x* is given by:

$$f_1(x,v) = n_{1S}\sqrt{\frac{m_e}{2\pi kT_e}}\exp\left(-\frac{m_e v^2}{2kT_e}\right)\exp\left(\frac{e_0\Phi(x)}{kT_e}\right)\mathrm{H}\left(-v-\sqrt{\frac{2e_0\Phi(x)}{m_e}}\right). \qquad (1)$$

A positive ion that has almost reached the collector, but that has been repelled with very small initial velocity at the collector, will have at the position *x*, where the potential is $\Phi(x)$ the velocity $\sqrt{2e_0(\Phi_C - \Phi(x))/m_i}$ in the direction of positive *x* axis towards the source. So the ion distribution at the position *x* is given by:

$$f_i(x,v) = n_{iS}\sqrt{\frac{m_i}{2\pi kT_i}}\exp\left(-\frac{m_i v^2}{2kT_i}\right)\exp\left(-\frac{e_0\Phi(x)}{kT_i}\right)\mathrm{H}\left(-v+\sqrt{\frac{2e_0(\Phi_C-\Phi(x))}{m_i}}\right). \qquad (2)$$

An emitted electron that has almost reached the source, but has then been repelled with a very small initial velocity, will have at the position *x*, where the potential is $\Phi(x)$ the velocity $\sqrt{2e_0\Phi(x)/m_e}$ in the direction of negative *x* axis towards the collector. So the distribution of the emitted electrons at the position *x* is given by:

$$f_2(x,v) = n_{2C}\sqrt{\frac{m_e}{2\pi kT_2}}\exp\left(-\frac{m_e v^2}{2kT_2}\right)\exp\left(\frac{e_0(\Phi(x)-\Phi_C)}{kT_2}\right)\mathrm{H}\left(v+\sqrt{\frac{2e_0\Phi(x)}{m_e}}\right). \qquad (3)$$

The meaning of the symbols is standard: m_i is the ion mass, m_e is the electron mass, e_0 is elementary charge, $\mathrm{H}(y)$ is the Heaviside unit step function, *k* is the Boltzmann constant, T_i, T_e

and T_2 are temperatures of the respective particle species and Φ_C is potential of the collector. The particle densities and fluxes of the particle species are given by:

$$n_1(x)=\int_{-\infty}^{\infty} f_1(x,v)\,dv,\ \Gamma_1=\int_{-\infty}^{\infty} vf_1(x,v)\,dv,\ n_i(x)=\int_{-\infty}^{\infty} f_i(x,v)\,dv,\ \Gamma_i=\int_{-\infty}^{\infty} vf_i(x,v)\,dv,$$
$$n_2(x)=\int_{-\infty}^{\infty} f_2(x,v)\,dv,\ \Gamma_2=\int_{-\infty}^{\infty} vf_1(x,v)\,dv. \tag{4}$$

Since the positive ions are singly charged, the collector is electrically floating when the following relation is fulfilled:

$$|\Gamma_1|=|\Gamma_i|+|\Gamma_2|. \tag{5}$$

The potential profile is given by the Poisson equation:

$$\frac{d^2\Phi}{dx^2}=-\frac{e_0}{\varepsilon_0}\left[n_i(x)-n_1(x)-n_2(x)\right]. \tag{6}$$

Here ε_0 is permittivity of the free space. The following variables are introduced:

$$\lambda_D=\sqrt{\frac{\varepsilon_0 kT_e}{n_{1S}e_0^2}},\ c_0=\sqrt{\frac{2kT_e}{m_e}},\ z=\frac{x}{\lambda_D},\ u=\frac{v}{c_0},\ \mu=\frac{m_e}{m_i},\ \alpha=\frac{n_{iS}}{n_{1S}},$$
$$\varepsilon=\frac{n_{2C}}{n_{1S}},\ \tau=\frac{T_i}{T_e},\ \sigma=\frac{T_2}{T_e},\ \Psi=\frac{e_0\Phi}{kT_e},\ \Psi_C=\frac{e_0\Phi_C}{kT_e},\ J_1=\frac{|\Gamma_1|}{n_{1S}c_0}, \tag{7}$$
$$J_2=\frac{|\Gamma_2|}{n_{1S}c_0},\ J_i=\frac{|\Gamma_i|}{n_{1S}c_0},\ \mathcal{L}=\frac{L}{\lambda_D}.$$

The Poisson equation (6) gets the following form:

$$\frac{d^2\Psi}{dz^2}=-\frac{\alpha}{2}\exp\left(-\frac{\Psi(z)}{\tau}\right)\left(1+\operatorname{erf}\sqrt{\frac{\Psi_C-\Psi(z)}{\tau}}\right)+\frac{1}{2}\exp(\Psi(z))\operatorname{erfc}(\Psi(z))+$$
$$+\frac{\varepsilon}{2}\exp\left(\frac{\Psi(z)-\Psi_C}{\sigma}\right)\left(1+\operatorname{erf}\sqrt{\frac{\Psi(z)}{\sigma}}\right)=-\rho(\Psi(z)). \tag{8}$$

Here $\rho(\Psi(z))$ is the normalized space charge density, erf(y) is the error function and erfc(y) is the complementary error function. They are given by:

$$\operatorname{erf}(y)=\frac{2}{\sqrt{\pi}}\int_0^y \exp(-t^2)\,dt=1-\operatorname{erfc}(y),\ \operatorname{erfc}(y)=\frac{2}{\sqrt{\pi}}\int_y^{\infty}\exp(-t^2)\,dt. \tag{9}$$

The floating condition (5) reads:

$$1-\alpha\sqrt{\mu\tau}\exp\left(-\frac{\Psi_C}{\tau}\right)-\varepsilon\sqrt{\sigma}\exp\left(-\frac{\Psi_C}{\sigma}\right)=0. \tag{10}$$

It is assumed that the length of the system $\mathcal{L}$ is big enough, that at a certain position z_0 the plasma is neutral. The potential at this position is $\Psi(z=z_0)=\Psi_P$. So:

$$\left.\frac{d^2\Psi}{dz^2}\right|_{\Psi=\Psi_P} = -\alpha\exp\left(-\frac{\Psi_P}{\tau}\right)\left(1+\operatorname{erf}\sqrt{\frac{\Psi_C-\Psi_P}{\tau}}\right) - \\ +\exp(\Psi_P)\operatorname{erfc}(\Psi_P) + \varepsilon\exp\left(\frac{\Psi_P-\Psi_C}{\sigma}\right)\left(1+\operatorname{erf}\sqrt{\frac{\Psi_P}{\sigma}}\right) = 0. \tag{11}$$

Since the second derivative of the potential at $z = z_0$ is zero, the first derivative (electric field) is a constant, which can also be set to zero, so

$$\left.\frac{d\Psi}{dz}\right|_{\Psi=\Psi_P} = 0. \tag{12}$$

Since the first and the second derivative of the potential at $z = z_0$ are both zero, z_0 is the inflection point of the potential. Taking into account the relation

$$\frac{1}{2}\frac{d}{dz}\left(\frac{d\Psi}{dz}\right)^2 = \frac{d^2\Psi}{dz^2}\frac{d\Psi}{dz}, \tag{13}$$

The Poisson equation (8) is multiplied by $d\Psi/dz$ and integrated over the potential in the form:

$$\int d\left(\frac{d\Psi}{dz}\right)^2 = \left(\frac{d\Psi}{dz}\right)^2 = 2\int\frac{d^2\Psi}{dz^2}d\Psi. \tag{14}$$

The integration is performed 2 times. First integration goes from the collector, $\Psi(z=0)=\Psi_C$, to the inflection point $\Psi(z=z_0) = \Psi_P$ and the second time it goes from the inflection point to the source, $\Psi(z=\mathcal{L}) = 0$. The gives the electric field condition at the collector:

$$2\int_{\Psi_C}^{\Psi_P}\frac{d^2\Psi}{dz^2}d\Psi = \left(\frac{d\Psi}{dz}\right)^2_{\Psi_P} - \left(\frac{d\Psi}{dz}\right)^2_{\Psi_C} = 0-\eta_C^2 = \\ = \frac{2}{\sqrt{\pi}}\left(\sqrt{\Psi_P}-\sqrt{\Psi_C}\right) - \exp(\Psi_C)\operatorname{erfc}\sqrt{\Psi_C} + \exp(\Psi_P)\operatorname{erfc}\sqrt{\Psi_P} - \\ -\alpha\tau\left[\exp\left(-\frac{\Psi_C}{\tau}\right)\left(1+2\sqrt{\frac{\Psi_C-\Psi_P}{\pi\tau}}\right) - \exp\left(-\frac{\Psi_P}{\tau}\right)\left(1+\operatorname{erf}\sqrt{\frac{\Psi_C-\Psi_P}{\tau}}\right)\right] + \\ +\frac{\varepsilon\sigma}{\sqrt{\pi}}\left[2\left(\sqrt{\Psi_C}-\sqrt{\Psi_P}\right) + \sqrt{\frac{\pi}{\sigma}}\exp\left(\frac{\Psi_P}{\sigma}\right)\left(1+\operatorname{erf}\sqrt{\frac{\Psi_P}{\sigma}}\right) + \sqrt{\frac{\pi}{\sigma}}\exp\left(\frac{\Psi_C}{\sigma}\right)\left(\operatorname{erfc}\sqrt{\frac{\Psi_C}{\sigma}}-2\right)\right]. \tag{15}$$

The second integral gives electric field condition at the source:

$$2\int_{\Psi_P}^{0}\frac{d^2\Psi}{dz^2}d\Psi = \left(\frac{d\Psi}{dz}\right)^2_{\Psi=0} - \left(\frac{d\Psi}{dz}\right)^2_{\Psi_P} = \eta_S^2 - 0 = 1 - 2\sqrt{\frac{\Psi_P}{\pi}} - \exp(\Psi_P)\operatorname{erfc}\sqrt{\Psi_P} - \\ -\alpha\tau\left[\begin{array}{l}\exp\left(-\frac{\Psi_P}{\tau}\right)\left(1+\operatorname{erf}\sqrt{\frac{\Psi_C-\Psi_P}{\tau}}\right) - 1 - \operatorname{erf}\sqrt{\frac{\Psi_C}{\tau}} + \\ + \dfrac{2\exp\left(-\frac{\Psi_C}{\tau}\right)\left(\Psi_P-\Psi_C+\sqrt{\Psi_C(\Psi_C-\Psi_P)}\right)}{\sqrt{\pi\tau(\Psi_C-\Psi_P)}}\end{array}\right] + \\ +\varepsilon\sigma\exp\left(-\frac{\Psi_C}{\sigma}\right)\left[1+2\sqrt{\frac{\Psi_P}{\pi\sigma}} + \exp\left(\frac{\Psi_P}{\sigma}\right)\left(\operatorname{erfc}\sqrt{\frac{\Psi_P}{\sigma}}-2\right)\right]. \tag{16}$$

In Fig. 1 an example of numerical solution of the Poisson equation (8) is shown. The following parameters are selected: μ =13670.48, α = 8.0, σ = 0.05, ε = 25.0 and τ = 1.0.

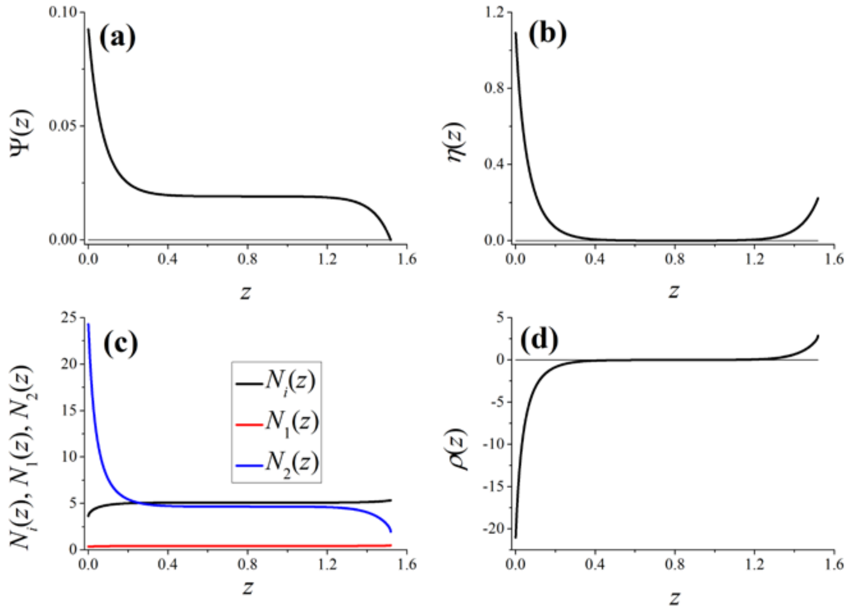

Figure 1: Numerical solution of the Poisson equation (8).

The solution of the system (10), (11), (15), (16) is Ψ_C = 0.092464, Ψ_P = 0.019112, η_C = 1.09141, η_S = 0.222459. Ψ_C is used as the first boundary condition and η_C as the second boundary condition. Potential profile $\Psi(z)$ is shown in plot (a), electric field profile $\eta(z)$ in plot (b) and space charge density profile $\rho(z)$ in graph (d). Potential decreases monotonically from the collector to the source, while space charge density increases. Inflection point of the potential is identified at $z = z_0$ = 0.8066. Potential at this point is 0.01911, which means perfect matching with the solution of the model Ψ_P = 0.019112. The integration of the Poisson equation breaks down at $z = \mathcal{L}$ = 1.5179. Electric field at this point is 0.22244. This is also in perfect agreement with η_S = 0.222459.

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