The effect of parallel magnetic field fluctuations on pressure driven MHD instabilities in toroidal plasmas

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Analytic derivation and numerical calculation of pressure driven MHD instabilities in toroidal plasmas is notoriously difficult. Instability is determined by apparently weak effects in the toroidal metric tensor. There are now new generations of gyro-kinetic codes that are being deployed to model MHD instabilities. Some of these codes are, or have been, only partially electromagnetic. It is usually assumed that the perturbed vector potential is parallel to the equilibrium field (so that the perturbed parallel magnetic field is nearly zero), although in some codes an imposed effective adiabatic parallel magnetic field is adopted. Similar such reduced models are also assumed in some non-linear MHD codes deployed for the study of edge localised modes. These codes also sometimes approximate the equilibrium toroidal magnetic field, by neglecting the magnetic well associated with fine pressure. The present contribution, offered in detail in Ref. [1], investigates the impact of code-relevant models for the parallel magnetic field and the equilibrium magnetic field on pressure driven instabilities in axisymmetric toroidal equilibria.

The Perturbed Curvature

From the unperturbed full momentum equation, assuming weak growth rate and isotropic pressure (neglecting kinetic corrections as discussed above), the curvature is

$$\kappa(t,x) = \frac{\nabla P}{B^2} + \frac{\nabla_\perp B}{B}.$$

Linearising this, we have

$$\delta\kappa = \frac{\nabla \delta P}{B^2} + \frac{\nabla_\perp \delta B_\parallel}{B} - 2\frac{\delta B_\parallel}{B}\kappa - \frac{\delta B_\perp (b\cdot\nabla)B + b(\delta B_\perp \cdot \nabla)B}{B^2}.$$

where only quantities assigned with δ are non-equilibrium. Considering flute modes, we expect $(b\cdot\nabla)\delta \sim \varepsilon \nabla_\perp \delta$ or slower. Variation of equilibrium quantities also scale linearly with ε. Hence taking $\delta P = -\xi^\psi dP/d\psi$ (collisionless ideal MHD),

$$\delta\kappa = \nabla_\perp \left[\frac{\delta B_\parallel}{B} - \xi^\psi \frac{1}{B^2}\frac{dP}{d\psi}\right](1+O(\varepsilon)). \quad (1)$$

Meanwhile under the MHD model we have exactly

$$\delta B_{\parallel} = -B(\nabla\cdot\xi_{\perp} + 2\kappa\cdot\xi_{\perp}) + B^{-1}\xi_{\perp}\cdot\nabla P. \quad (2)$$

Finally, the *convenient* form for the potential energy in the plasma, on deploying Eq. (2) can be written as,

$$\delta W_{\perp} = \frac{1}{2}\int_P d^3x \left[|\delta B_{\perp}|^2 + B^2\left|\delta B_{\parallel} - B^{-1}\xi_{\perp}\cdot\nabla P\right|^2 - 2(\xi_{\perp}\cdot\nabla P)(\kappa\cdot\xi^*_{\perp}) - J_{\parallel}(\xi^*_{\perp}\times e_{\parallel})\cdot\delta B_{\perp}\right]. \quad (3)$$

Hence the stabilising second term is seen to be associated with fluctuations in total pressure (magnetic + thermal), or from Eq. (1), the fluctuating magnetic curvature.

Analytic Solutions for long wavelength modes

We now follow [2] assuming a local large aspect ratio expansion. We identify specifically the effect of inconsistent $\delta B_{\parallel}$ and the equilibrium toroidal field. Let $B = F\nabla\phi + \nabla\phi\times\nabla\psi$ with

$$F = F_0 + \sigma_2 F_2(\psi)$$

where $F_0 = R_0 B_0$ is a constant and

$$\frac{dF_2}{dr} = B_0\left[\frac{\alpha}{2q^2} - \frac{\varepsilon}{q^2}(2-s)\right], \quad \alpha \equiv -\frac{2q^2R_0}{B_0^2}\frac{dP}{dr}.$$

Here $\sigma_2 = 1$ if the equilibrium field is consistently taken into account, or $\sigma_2 = 0$ in the case where the effect is neglected. If we also allow for a model for $\delta B_{\parallel}$, or δB^{ϕ} we obtain δW_{RED} (reduced) relative to δW_{CON} (consistent):

$$\delta W_{RED} = \delta W_{CON} + 2\pi^2 B_0^2 R_0^{-1}\int_0^a dr\, r\left(\hat{\xi}_0^r\right)^2\mathcal{Q}, \quad (4)$$

$$\mathcal{Q} = \left[\frac{R\hat{\delta B}^{\phi}}{\hat{\xi}_0^r B_0} + \frac{V(r)}{2}\right]^2 + (\sigma_2 - 1)\frac{1}{B_0}\frac{dF_2}{dr}\left[\frac{R\hat{\delta B}^{\phi}}{\hat{\xi}_0^r B_0} + \frac{V(r)}{2} - \varepsilon\frac{n}{m}\left(\frac{n}{m} - \frac{1}{q}\right)\right], \quad (5)$$

$$V(r) = \frac{\alpha}{q^2} + 2\varepsilon\left[\left(\frac{n}{m} + \frac{2}{q}\right)\left(\frac{n}{m} - \frac{1}{q}\right) + \frac{s}{q^2}\right]. \quad (6)$$

We may lift known expressions for δW_{CON}, e.g. the internal kink mode, infernal modes, interchange and ballooning. For basic gyrokinetic or fluid codes which neglect $\delta B_{\parallel}$ entirely without compensation, but keep realistic equilibrium toroidal field $\sigma_2 = 1$, one obtains an unphysical absolutely stabilising effect associated with $\mathcal{Q} = (V/2)^2$. This effect can be attributed to perturbations in the curvature, which for a self-consistent case should be zero.

Solutions for codes that neglect $\delta B_{\parallel}$ and approximate toroidal equilibrium field

In this section we let

$$\delta B^{\phi} = \sigma_1\left(-\frac{\xi_0^r B_0 V(r)}{2R}\right) \quad (7)$$

such that $\sigma_1 = 1$ for a self-consistent MHD case, and $\sigma_1 = 0$ if $\delta B_\parallel$ effects are neglected.

Infinite n Ballooning Modes can be treated with the treatment here:

$$\frac{\partial}{\partial\theta}\left[\left\{1+(s\theta-\alpha\sin\theta)^2\right\}\frac{\partial}{\partial\theta}\xi^r\right]$$
$$+\alpha\left[\cos\theta-\varepsilon\left\{1-\frac{\sigma_2}{q^2}\right\}+\sin\theta(s\theta-\alpha\sin\theta)\right]\xi^r-(\sigma_2-\sigma_1)\frac{\alpha^2}{4q^2}\xi^r=0. \quad (8)$$

The effect of artificially setting $\sigma_1 = 0$ and/or $\sigma_2 = 0$ is weak for traditional ballooning modes in tokamaks which are typically of concern in the edge where $q \gg 1$ and $\alpha \sim 1$. However, for Mercier modes, the approximation is important.

Interchange Modes can be developed robustly in the long wavelength limit, or for short wavelengths. One obtains the modified Mercier criterion for instability:

$$\alpha\left[\varepsilon\left(\frac{\sigma_2}{q^2}-1\right)-\frac{(\sigma_2-\sigma_1)}{4q^2}\alpha\right]>\frac{s^2}{4}. \quad (9)$$

Hence, one recovers the usual MHD results for the self-consistent parameterisation $\sigma_1 = \sigma_2 = 1$. But, for reduced MHD codes that deploy $\sigma_1 = \sigma_2 = 0$, Mercier modes are unstable for $-4\varepsilon\alpha > s^2$. It is seen that the drive for interchange modes in a cylinder has disappeared, and one is left with only the additional factor that arrives from toroidal curvature. One can therefore only obtain interchange modes for cases where α is negative (sometimes relevant for plasmas that have accumulated impurities in the core)

Infernal Modes are described by the Eigenvalue equation [3]:

$$\frac{\gamma^2}{\omega_A^2}=\frac{n^2}{\Lambda}\left\{\frac{2^{-3}\alpha^2\Lambda_{m,n}}{(m+1)^2(m+2)}+\frac{2^{-1}}{(m+1)(m+3)}\left[\varepsilon\alpha\left(\frac{\sigma_2}{q_r^2}-1\right)-(\sigma_2-\sigma_1)\frac{\alpha^2}{4q_r^2}\right]-\left(\frac{\Delta q}{q_r}\right)^2\right\}, \quad (10)$$

where $\Lambda = 1+2q^2$ in the ideal MHD limit, and $\Lambda_{m,n}$ is a coefficient related to infernal mode coupling [4].

Internal Kink Mode stability is also easily established. On setting $\sigma_2 = \sigma_1 = 0$ we obtain

$$\delta\hat{W}=\beta_p+(1-q_0)\left[\frac{13}{48}-3\beta_p^2\right], \quad \beta_p=-\frac{2}{B_p^2r^2}\int_0^r dr\, r^2\frac{dP}{dr}\sim 1, \quad (11)$$

where the term $(1-q_0)\left[13/48-\beta_p^2\right]$ is the well known self-consistent Bussac [2], but it is smaller than the artificial term η_p. For the reduced problem, the internal kink will be stable if the pressure is peaked in the core, or unstable if the pressure is hollow. Codes with $\sigma_1 = \sigma_2 = 0$ that seek to obtain an internal kink mode may wish to consider a hollow pressure profile.

Improvement for Partially Electromagnetic Gyrokinetic Codes

Some gyrokinetic codes impose an **effective** $\delta B_\parallel$ which recovers the essential finite beta effects associated with $\delta B_\parallel$. These codes impose the approximate form $\delta\dot{B}_\parallel = B^{-3}(\nabla\Phi\times B)\cdot$

∇P, where Φ is the perturbed vector potential. This approximate form is equivalent $\delta B_{\|} = B^{-1}\xi_{\perp} \cdot \nabla P$. This is (to leading order in gyro-radius) achieved via the dual transformations $\Omega_{\nabla B} \rightarrow \Omega_{\kappa}$ and $\delta B_{\|} \rightarrow 0$, where

$$\Omega_{\nabla B} = -i\frac{(b \times \nabla B) \cdot \nabla}{B} \text{ and } \Omega_{\kappa} = -i(b \times \kappa) \cdot \nabla.$$

The effects of $\delta B_{\|}$ appear in gyrokinetic codes to leading order in gyroradius in terms involving K:

$$K = i\delta \dot{B}_{\|} - \Omega_{\nabla B}\Phi. \tag{12}$$

So, the approximation deployed in gyrokinetic codes conveniently yield $K = -\Omega_{\kappa}\Phi$. For pressure driven instabilities, the effective $\delta B_{\|}$ nearly gives correct result. One does observe an error for internal kink modes - the hardest of pressure driven instabilities to get correct.

The MHD calculations undertaken yield a more accurate solution to $\delta B_{\|}$, identified from condition $\delta W_{RED} = \delta W_{CON}$, i.e. $\mathscr{Q} = 0$. From Eq. (4), for a physical magnetic field $\sigma_2 = 1$, this requires that $\delta B^{\phi} = -\xi_0^r B_0 V(r)/(2R)$ (as also seen from the consistent case $\sigma_2 = 1$ in Eq. (7)). The corresponding parallel magnetic field is easily seen to be:

$$\delta B_{\|} = -B_0 \frac{\xi_0^r}{R} \left\{ \frac{\alpha}{2q^2} + \varepsilon \left[\left(\frac{n}{m} + \frac{1}{q} \right) \left(\frac{n}{m} - \frac{1}{q} \right) \right] \right\}. \tag{13}$$

By transforming from MHD variables to gyrokinetic variables, we can now obtain an improvement to the transformation undertaken in gyro-kinetic codes. In particular gyrokinetic codes will be capable of recovering ideal MHD pressure driven instabilities with the following transformation:

$$\delta B_{\|} \rightarrow 0, \quad \Omega_{\nabla B} \rightarrow \Omega_{\kappa} + \frac{2i}{qRB} B \cdot \nabla \tag{14}$$

The correction in this expression contains the magnetic operator $B \cdot \nabla$. The internal kink mode is a special case, since the $B \cdot \nabla$ operation on an $m = 1$ perturbation is not necessarily small far from the rational surface.

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