

Modelling of the non-local transport of energy in laser plasmas with high-order numerical methods

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Abstract

The description of the energy transport processes in the laser plasma is crucial for capturing the dynamics of the laser–target interaction relevant to shock ignition [1] and pre-pulses of ultra-intense lasers [2]. The diffusion approximation of the heat and radiation transport become inadequate even for the laser intensities $\lesssim 10^{15}$ W/cm² in many cases [3]. The non-local nature of the transport phenomena must be considered due to long mean-free-paths of the heated electron species compared to the characteristic length of the plasma temperature variations. The shift in the physical models of plasmas must be reflected in the numerical treatment of the problem too. The high-order finite element methods present a favourable option. We have proposed such method recently [4] and continue in the effort towards better modelling and understanding of the non-local phenomena by means of numerical simulations.

Introduction

Hydrodynamic description presents a compelling option for description of laser plasma. However, the fluid framework can not capture the details of transport phenomena going beyond the local diffusion theory [3]. An effort is then made to overcome this limitation by formulation of a closure model reflecting the kinetic nature of the species.

The transport phenomena in plasmas are well-described in the fluid approximation by Vlasov equation, which models the kinetics of the species with an appropriate collision operator. In order to formulate a numerically efficient closure model, we restrict ourselves to Bhatnagar–Gross–Krook (BGK) collision operator [5] as used recently in [4]. The kinetic

equation for electrons then reads:

$$\frac{\partial f}{\partial t} + \vec{v} \nabla f + \frac{q_e}{m_e} (\vec{E} + \vec{v} \times \vec{B}) \cdot \nabla_{\vec{v}} f = \nu_e (f_M - f) + \frac{\nu_{ei} + \nu_e}{2} \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial f}{\partial \mu}, \quad (1)$$

where $f = f(\vec{x}, \vec{v}, t)$ is the distribution function of the electrons in the phase space at space coordinate \vec{x} , velocity \vec{v} and time t . The directional cosine of the velocity is denoted as μ . The vector fields \vec{E} and \vec{B} are the electric and magnetic field respectively. The constants q_e , m_e denote the charge of electron and its mass, whereas ν_e and ν_{ei} are the electron-electron and electron-ion collision frequencies respectively.

For the long timescales of interest in hydrodynamic simulations, the transport can be considered as stationary and the assumption of low anisotropy is justified as well. The Cartesian expansion of the distribution to the first order in form $f = f_0 + \vec{n} \cdot \vec{f}_1$ gives [6]:

$$\lambda_e^{-1} (f_M - f_0) = \frac{1}{3} \nabla \cdot \vec{f}_1 + \frac{1}{3} \frac{q_e}{m_e} \vec{E} \cdot \left(\frac{1}{v} \frac{\partial \vec{f}_1}{\partial v} + \frac{2}{v^2} \vec{f}_1 \right), \quad (2)$$

$$-(\lambda_e^{-1} + \lambda_{ei}^{-1}) \vec{f}_1 = \nabla f_0 + \frac{q_e}{m_e} \vec{E} \frac{1}{v} \frac{\partial f_0}{\partial v} + \frac{q_e}{m_e} \frac{1}{v} \vec{B} \times \vec{f}_1, \quad (3)$$

where $\lambda_e = \nu / \nu_e$, $\lambda_{ei} = \nu / \nu_{ei}$. Considering only the case without magnetic field in the following, the non-local Ohm's law can be obtained by integrating (3) over velocity as:

$$\vec{j} = \frac{4\pi}{3} q_e \int \lambda_s \left(\frac{\partial f_0}{\partial v} \vec{\mathcal{E}} + v \nabla f_0 \right) v^2 dv \stackrel{\lambda \sim v^4}{\simeq} -8\pi q_e \left(\int \lambda_s f_0 v dv \right) \vec{\mathcal{E}} + \frac{4\pi}{3} q_e \nabla \int \lambda_s f_0 v^3 dv, \quad (4)$$

where the symbols $\lambda_s = 1/(\lambda_e^{-1} + \lambda_{ei}^{-1})$, $\vec{\mathcal{E}} = \frac{q_e}{m_e} \vec{E}$ were introduced.

The quasi-neutrality constraint of the classical hydrodynamics $\nabla \cdot \vec{j} = 0$ considering only irrotational currents results in the zero current condition $\vec{j} = 0$ giving the explicit formula for the electric field:

$$\vec{\mathcal{E}} = \frac{\nabla \int f_0 v^7 dv}{6 \int f_0 v^5 dv}. \quad (5)$$

Numerical model

The approach to numerical solution of the system (2-3) is to solve it on the spatial domain and performing outer integration over the velocity space from $7v_{th}$ (v_{th} is the electron thermal velocity), where the f_0 and \vec{f}_1 are set equal zero, to zero (decelerating P1). For this purpose, the fully implicit backward Euler scheme is used. The next step of the procedure is calculation of the electric field according to (5), which appears non-linearly in the system, so Picard iterations are performed to obtain the implicit self-consistent electric field $\vec{\mathcal{E}}$. Moreover, the temperatures are updated implicitly, where the source terms are linearised reducing the computation overhead [7]:

$$f_M(T^{n+1}) = \frac{\partial f_M}{\partial T} (T^{n+1} - T^n) + f_M(T^n) = S_A T^{n+1} + S_b. \quad (6)$$

Returning back to the solution on the spatial domain, finite element method (FEM) is applied enabling setting arbitrary polynomial order of the elements and making the formulation compatible with the high-order curvilinear hydrodynamic code PETE [8]. The following choices of the function spaces are made on the computational domain Ω :

- discontinuous temperatures: $T_e \approx \boldsymbol{\varphi} \cdot \mathbf{T}_e$ $\boldsymbol{\varphi}_i \in L_2(\Omega) \quad \forall i$
- discontinuous f_0 distribution: $f_0(v) \approx \boldsymbol{\psi} \cdot \mathbf{f}_0(v)$ $\boldsymbol{\psi}_i \in L_2(\Omega) \quad \forall i$
- Raviart-Thomas \vec{f}_1 distribution: $\vec{f}_1(v) \approx \Upsilon \cdot \mathbf{f}_1(v)$ $\Upsilon_i \in H_{div}(\Omega) \quad \forall i$
- Nedelec electric field: $\vec{\mathcal{E}} \approx \boldsymbol{\xi} \cdot \mathbf{E}$ $\boldsymbol{\xi}_i \in H_{curl}(\Omega) \quad \forall i$

Next, the discrete matrices are constructed as follows:

$$\mathbb{M}_{\lambda_e} = \int_{\Omega} \lambda_e^{-1} \boldsymbol{\psi} \otimes \boldsymbol{\psi}^T d\Omega, \quad \mathbb{M}_{\lambda_s} = \sum_i \int_{\Omega} \lambda_s^{-1} \Upsilon_{i,\bullet} \otimes \Upsilon_{i,\bullet}^T d\Omega, \quad (7)$$

$$\mathbb{D} = \int_{\Omega} \nabla \cdot \Upsilon \otimes \boldsymbol{\psi}^T d\Omega, \quad \mathbb{B}_{\partial\Omega} = \int_{\partial\Omega} \text{Tr} \boldsymbol{\psi} \otimes (\Upsilon^T \cdot \vec{n}) d\Omega, \quad (8)$$

$$\mathbb{M}_{\vec{\mathcal{E}}} = \int_{\Omega} \vec{\mathcal{E}} \cdot \Upsilon \otimes \boldsymbol{\psi}^T d\Omega, \quad \mathbb{M}_{SA} = \int_{\Omega} \lambda_e^{-1} S_A \boldsymbol{\varphi} \otimes \boldsymbol{\psi}^T d\Omega, \quad (9)$$

$$\mathbf{b}_{Sb} = \int_{\Omega} \lambda_e^{-1} S_b \boldsymbol{\psi}^T d\Omega. \quad (10)$$

The discrete form of the system (2,3) decelerating in velocities with the step Δv can be written as ($\mathbf{f}^- = \mathbf{f}(v - \Delta v)$, $\mathbf{f}^+ = \mathbf{f}(v)$):

$$\mathbb{M}_{\lambda_e} \mathbf{f}_0^- + \frac{1}{3} \mathbb{D} \mathbf{f}_1^- + \frac{1}{3v} \left(\frac{2}{v} - \frac{1}{\Delta v} \right) \mathbb{M}_{\vec{\mathcal{E}}} \mathbf{f}_1^- = -\frac{1}{3v} \frac{1}{\Delta v} \mathbb{M}_{\vec{\mathcal{E}}} \mathbf{f}_1^+ + \mathbb{M}_{SA} \mathbf{T}_e + \mathbf{b}_{Sb}, \quad (11)$$

$$\mathbb{M}_{\lambda_s} \mathbf{f}_1^- + (\mathbb{B}_{\partial\Omega} - \mathbb{D}^T) \mathbf{f}_0^- - \frac{1}{v} \frac{1}{\Delta v} \mathbb{M}_{\vec{\mathcal{E}}}^T \mathbf{f}_0^- = -\frac{1}{v} \frac{1}{\Delta v} \mathbb{M}_{\vec{\mathcal{E}}}^T \mathbf{f}_0^+, \quad (12)$$

eliminating \mathbf{f}_0^- yields:

$$\begin{aligned} & \left(\mathbb{M}_{\lambda_s} + \left(-\mathbb{B}_{\partial\Omega} + \mathbb{D}^T + \frac{1}{v} \frac{1}{\Delta v} \mathbb{M}_{\vec{\mathcal{E}}}^T \right) \mathbb{M}_{\lambda_e}^{-1} \left(\frac{1}{3} \mathbb{D} + \frac{1}{3v} \left(\frac{2}{v} - \frac{1}{\Delta v} \right) \mathbb{M}_{\vec{\mathcal{E}}} \right) \right) \mathbf{f}_1^- = \\ & = -\frac{1}{v} \frac{1}{\Delta v} \mathbb{M}_{\vec{\mathcal{E}}}^T \mathbf{f}_0^+ + \left(-\mathbb{B}_{\partial\Omega} + \mathbb{D}^T + \frac{1}{v} \frac{1}{\Delta v} \mathbb{M}_{\vec{\mathcal{E}}}^T \right) \mathbb{M}_{\lambda_e}^{-1} \left(-\frac{1}{3v} \frac{1}{\Delta v} \mathbb{M}_{\vec{\mathcal{E}}} \mathbf{f}_1^+ + \mathbb{M}_{SA} \mathbf{T}_e + \mathbf{b}_{Sb} \right). \end{aligned} \quad (13)$$

Finally, the matrix can be rearranged to a more convenient form giving better insight into the properties of the matrix (disregarding the boundary terms $\mathbb{B}_{\partial\Omega}$ for the moment):

$$\underbrace{\left(\mathbb{M}_{\lambda_s} + \frac{1}{3} \mathbb{D}^T \mathbb{M}_{\lambda_e}^{-1} \mathbb{D} \right)}_{\text{SPD}} + \underbrace{\left(\frac{1}{3v^2 \Delta v} \left(\frac{2}{v} - \frac{1}{\Delta v} \right) \mathbb{M}_{\vec{\mathcal{E}}}^T \mathbb{M}_{\lambda_e}^{-1} \mathbb{M}_{\vec{\mathcal{E}}} \right)}_{\text{symmetric (PD when } \Delta v > v/2)} + \underbrace{\left(\frac{2}{3v^2} \mathbb{D}^T \mathbb{M}_{\lambda_e}^{-1} \mathbb{M}_{\vec{\mathcal{E}}} \right)}_{\text{asymmetric}}. \quad (14)$$

The system is then hybridized introducing Lagrange multipliers γ at the skeleton of the computation mesh imposing weak continuity of the \vec{f}_1 elements [9]. This procedure greatly reduces the size of the problem approximately by factor of the polynomial order of the elements and can be fully parallelized. The resulting linear system for γ is solved by standard algorithms for non-symmetric sparse matrices.

Conclusions

An efficient numerical scheme based on high-order finite elements for $P1$ transport with BGK collision operator was proposed. Unlike the Sn model proposed in [4], an explicit formula (5) for the self-consistent electric field is found and the energy spectrum of the electrons is fully resolved. However, the properties of the matrix (14) originating from $P1$ model itself may deteriorate the numerical solution. Finally, a great increase of computational efficiency was achieved by performing hybridization of the discrete formulation.

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