Towards a universal collision-free sheath solution with warm ions

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By numerically solving the collision-free plane-discharge problem with Maxwellian electrons and a neutral Maxwellian ion source of arbitrarily high ion-source temperature, it has been found recently that the electric field E in both the plasma and sheath regions can be expressed in terms of the sheath potential φ via the equation $\varepsilon^2E^2/2+n_eT_e/2 = A\varphi$ (cf. Fig. 7 in Ref. [1]), with n_e, T_e the electron density and temperature, $A \approx \mathrm{e}^{-\varphi_s}/4$, and φ_s the plasma-edge potential. The idea of finding such a universal sheath solution in the form $\frac{\varepsilon^2E^2}{2} = \mathcal{P}(\varphi,T_n)$ will be further elaborated here as follows.

The reference sheath model corresponds to a collision-free discharge of length $2L$, located between two plane-parallel plates and symmetric with respect to the symmetry plane at $x = 0$. The normalized distance, ion and electron densities, velocities, temperatures, electrostatic potential, velocity distribution functions, and electric field are normalized as $\frac{x}{L} \leftrightarrow x$, $\frac{n_{i,e}}{n_0} \leftrightarrow n_{i,e}$, $\frac{v_i}{c_{se}} \leftrightarrow v_i$, $\frac{u_i}{c_{se}} \leftrightarrow u_i$, $\frac{T_i}{T_e} \leftrightarrow T_i$, $\frac{e\Phi}{kT_e} \leftrightarrow \Phi = -\varphi$, $\frac{c_{se}f_i}{n_0} \leftrightarrow f_i$, $\frac{LE}{kT_e/e} \leftrightarrow E$, and $E = -d\Phi/dx \leftrightarrow d\varphi/dx$, respectively, with e the positive elementary charge, k the Boltzmann constant, $c_{se} \equiv (kT_e/m_i)^{1/2}$ the zero-T_i ion-sound velocity, and m_i the ion mass. For completeness we also introduce the smallness parameter $\varepsilon \equiv \lambda_D/L$, where $\lambda_D = (\epsilon_0 kT_{e0}/n_0e^2)^{1/2}$ is the electron Debye length, with ϵ_0 the "vacuum permeability". The ion source of strength $S_i = Rn_nn_{e0}\mathrm{e}^{\beta\Phi/kT_e}\mathrm{e}^{-m_iv^2/2kT_n}/(2\pi kT_n)^{1/2}$ corresponds to the Bissell and Johnson (BJ) model [2], which in the limit $T_n \to 0$ reduces to the Tonks and Langmuir (TL) discharge where, according to Harrison and Thompson [3], β determines the source profile as a function of potential.

With these assumptions and definitions, the solutions of the Boltzmann equation for the ion VDFs in the quasineutral plasma region take, in the limit of cold ion sources and for warm ion sources with $T_n > 0.5$, the TL and BJ forms (for the latter see our accompanying paper P1.3002)

$$\begin{aligned} f_{i,TL}(v,\varphi) &= \frac{2\sqrt{2}}{\pi}\left[\frac{1}{2\sqrt{\varphi - v^2/2}} - F_D\left(\sqrt{\varphi - v^2/2}\right)\right], \\ f_{i,BJ}(v,\varphi) &= \frac{(2T_n+1)\mathrm{e}^{-(1+\frac{1}{2T_n})\frac{\varphi_b}{2}}}{2\pi T_n}\int_{\varphi'}\frac{\sqrt{\varphi_b-\varphi'}}{\sqrt{\varphi'}\sqrt{2(\varphi'-\varphi+v^2/2)}}\mathrm{e}^{-\varphi'+\frac{\varphi'-\varphi+v^2/2}{T_n}}d\varphi' \,, \end{aligned} \tag{1}$$

respectively, where $F_D(z)$ denotes the Dawson function. The ion densities, directional velocities, temperatures and higher velocity moments such as heat and energy fluxes have to be calculated as the (position-dependent) mth moments $\left\langle v_i^m \right\rangle = \int f_i v^m dv/n_i$. The electrons, on the other hand, are assumed to be Maxwellian-distributed in both the plasma and sheath regions with constant temperature $T_e = T_{e0} = 1$.

The above VDFs have been derived under the condition of strict plasma neutrality, $n_i - n_e = n_i - \mathrm{e}^{-\varphi} = 0$, which breaks at some point $\varphi_b < 1$ characterized by the electric-field singularity. The latter is identified as the plasma edge, $\varphi_b = \varphi_{PE}(T_n)$, at which both the unified Bohm criterion $u_{i,B}^2 = u_i^2(\varphi_{PE}) = 1 + \varkappa_{iPE}T_{iPE}$ and the condition $d(n_i - n_e)/d\varphi = 0$

are automatically satisfied. The plasma-edge potential φ_{PE}, ion temperature T_{iPE}, and ion polytropic coefficient $\varkappa_{iPE}$ are tabulated in Refs. [1, 4], where the quasi-analytic approximations to their dependencies on the source temperature and the particular β-profiles are presented as well.

At the plasma edge the expressions (1) reduce to the particular ones with $\varphi = \varphi_{PE}$, and in the region between the plasma edge and the wall the ion VDFs become functions of $v^2/2 - \Delta\varphi$, with $\Delta\varphi = \varphi - \varphi_{PE}$. While in the TL model the ion VDF in both the plasma and sheath regions is characterized by a single population with maximum velocities limited to $\sqrt{2\varphi}$, the situation in the BJ model is more complex (cf. our accompanying EPS paper 1.3002). At the plasma edge and in the sheath, however, the ion VDF reduces to only two populations, both characterized by $v > 0$, which we call "left" and "right" branches, as considered with respect to a non-differentiable VDF-maximum. In Fig. 1 we illustrate the ion VDFs at the PE and at four locations inside the sheath, with the marked areas (densities) belonging to the "left branches" at the first three potentials, indicating the fastest density drop between the first and second ones, i.e., in the proximity of the PE.

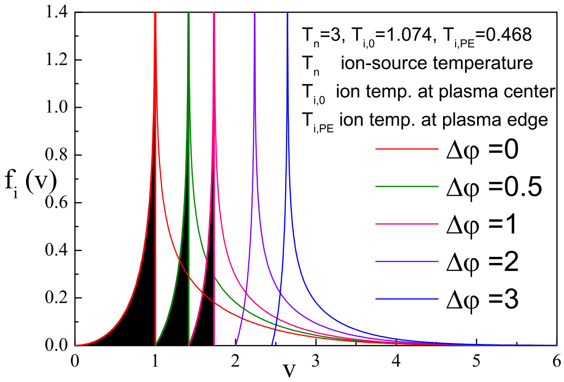

Figure 1: Illustration of BJ ion VDFs and the effect of acceleration of slow ions (the "left" branch) at the sheath edge on their density.

Note that the VDFs in Fig. 1 are normalized to unity at the PE rather than at the plasma center. Moreover, we set there $\varphi_{PE} = 0$ and hereinafter replace $\Delta\varphi$ with φ. This choice simplifies employing in the sheath analysis some additional VDFs which are not consistent with the quasi-neutral plasma models. Such VDFs are the Dirac delta distribution and the rectangular VDF, also known also as the "waterbag (WB)" VDF [5]:

$$f_{i,WB}(v,\varphi) = A[H(v - v_-) - H(v - v_+)], \text{ with } A = n_0/(v_{0+} - v_{0-}), \quad (2)$$

where $H(z)$ is the Heaviside step function and $v_\pm^2 = v_{0\pm}^2 + 2\varphi$, with the properties $n = A(v_+ - v_-)$, $u = (v_+ + v_-)/$, $T = (v_+ - v_-)^2/12$ and $\varkappa = 3$. At the sheath entrance the WB VDF must satisfy the conditions $n_i - n_e = 0$ and $d(n_i - n_e)/d\varphi = 0$. In this work we have revealed for the first time that the latter condition requires fulfillment of $v_{0+}v_{0-} = 1$. Thus, the conditions $v_{0+} > v_{0-} > 0$ and $v_{0+}v_{0-} = 1$ together can be regarded as the particular generalization of the classical Bohm criterion to warm adiabatic ions. In this context we note that in the limit $v_{0+} \to v_{0-} = v_0$ the waterbag distribution reduces to the Dirac delta distribution $A\delta(v - v_0)$ (for which $T_i = 0$). Increasing the temperature at the sheath entrance, however, requires v_{0-} to decrease towards zero. In the WB model with high temperature the ion velocity and thus, due to $u_i n_i = const$, also the ion density, is much more sensitive to the electric field than in the case of the self-consistent warm BJ model, cf. the densities represented by the color-filled areas below the left branches of the VDFs in Fig. 1.

We start investigating the sheath region by comparing the charge densities and their derivatives corresponding to all types of ion VDFs considered here, as illustrated in

Fig. 2(a). In Fig. 2(b) we illustrate the main quantity of interest here, i.e., the electric-field pressure which, according to the Poisson equation $n_i - n_e = d(\varepsilon^2 E^2/2)/d\varphi$, can be found from $\varepsilon^2 E^2/2 = \int (n_i - n_e) d\varphi$. It is indicative that the curves corresponding to the self-consistent (BJ and TL) models are similar, i.e., apparently identical to each other to within the accuracy of some hypothetical multiplicative constant which depends on the source temperature. Note that the waterbag and δ distributions exhibit qualitatively similar behavior but quantitatively depart considerably from the self-consistent ones. In previous works, the supposed multiplicative constant has been represented by the

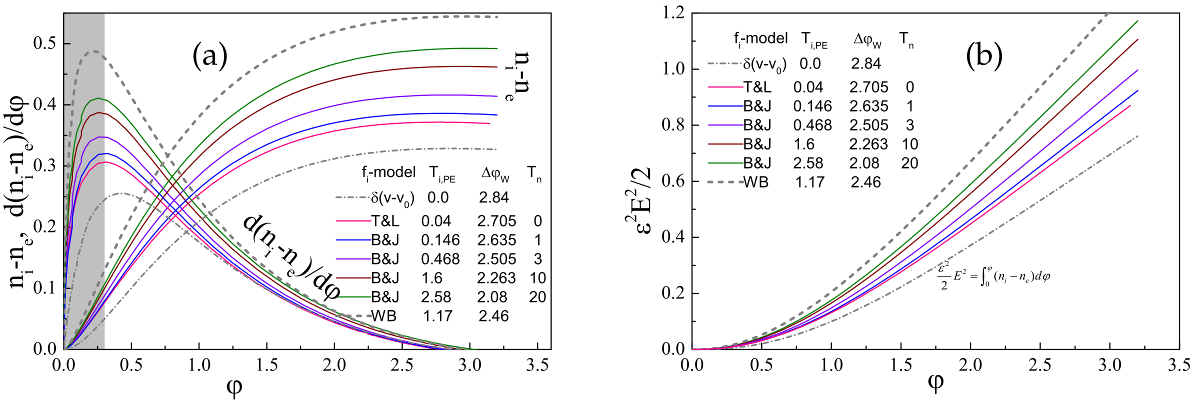

Figure 2: Comparisons of charge density and its derivative (a) and the corresponding electric-field-pressure density (b) within the sheath for the self-consistent and the artificial VDFs with cold and warm ions at the sheath entrance.

factor $\exp(-\varphi_{PE})$ referred to above. In the sheath formulation, however, where we choose the plasma-edge (i.e., sheath-entrance) potential to be zero and normalize the PE VDFs to unity, we do not expect dependence on φ_{PE} any more.

In order to shed more light onto functional dependence of $\varepsilon^2 E^2/2$ on φ in the present normalization we plot in Fig. 3(a), similarly as in Ref. [1], results from Fig. 2(b) in the form $\mathcal{E} = \varepsilon^2 E^2/2 + (n_{e0} - n_e)T_e/2$ (assuming $n_{e0} = 1$, $n_e = \mathrm{e}^{-\varphi}$ and $T_e = 1$). From Fig. 3(a), it emerges that also in the sheath normalization all curves apparently exhibit good

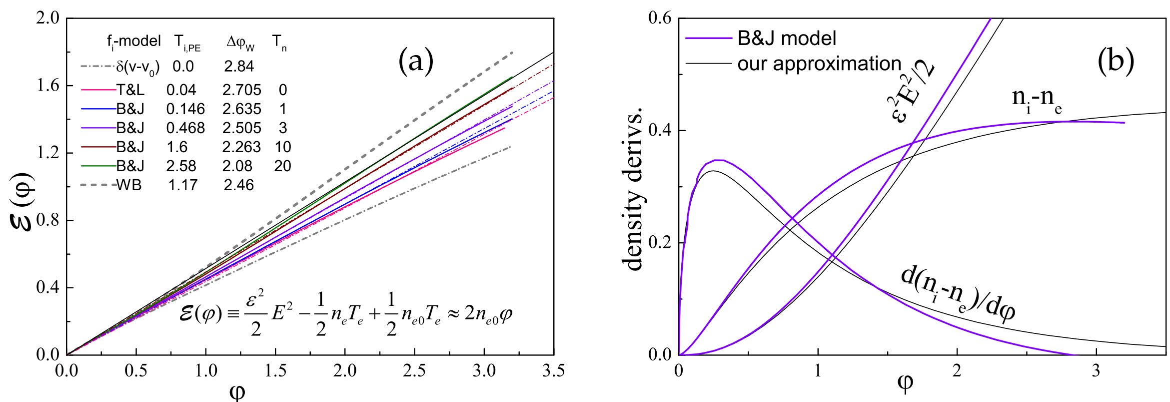

Figure 3: Representation of a particular combination of the electrostatic and the kinetic electron pressures which apparently exhibit, linear dependence on the sheath potential (a), and its reconstruction via the present trial function (b).

linearity (cf. the linear fitting functions sketched). However, we here leave open the question of finding a possible multiplicative factor by which these curves would become identical to each other. Instead, we simply approximate all of these curves by $\mathcal{E} =$

$C(T_n)\varphi$, taking the estimated value $C(T_n) = 2$ as a reference. The reference equation $\varepsilon^2E^2/2 = e^{-\varphi}/2+2\varphi-1/2$ turns out to fit well the first derivative $d(\varepsilon^2E^2/2)/d\varphi = (n_i - n_e)$ obtained, e.g., for $T_n = 3$ in Fig. 2(a), but not also the second one, $d^2(\varepsilon^2E^2/2)/d\varphi^2 = d(n_i - n_e)/d\varphi = e^{-\varphi}/2$. We adjust this behavior by multiplying the given solution by the factor $\mathrm{erf}(2\sqrt{\varphi})$ (where erf stands for the error function), so that the second derivative of the electric-field pressure becomes $d^2(\varepsilon^2E^2/2)/d\varphi^2 = \mathrm{erf}(2\sqrt{\varphi})e^{-\varphi}/2$. After integrating this expression we obtain

$$\frac{\varepsilon^2E^2}{2} = \frac{\sqrt{5}}{50}\mathrm{erf}(\sqrt{5\varphi})(10\varphi-11)+\frac{1}{5\sqrt{\pi}}e^{-5\varphi}\sqrt{\varphi}+\frac{1}{2}e^{-\varphi}\mathrm{erf}(\sqrt{2\varphi}) \equiv \mathcal{P}(\varphi). \quad (3)$$

In Fig. 3(b) we show the results of the electric-field pressure thus reconstructed and the corresponding derivatives, in comparison with the curves obtained for the representative case $T_n = 3$. Based on the relatively good agreement of the results one may conclude that the expression (3), rather than the above-mentioned expression from previous work, can serve as a reference one for finding the universal sheath solution that will be exact regarding other temperatures as well.

The last step here consists in finding sheath solution in the form $\varphi(x)$, starting from the definition of the electric field $d\varphi/dx = E$, with $E = \sqrt{2\mathcal{P}}/\varepsilon$, i.e., $x = \varepsilon\int d\varphi/\sqrt{2\mathcal{P}}$. Upon applying this method to the electric fields given by the expression (3) and finding the exact analytic field for the representative case $T_n = 3$, we obtain the respective sheath solution as presented in Fig. 4. It should be noted that the results do not depend on the wall potential φ_W (as obtained from equality of the ion and electron currents to the wall), which in Fig. 4 is given there just for illustration. Any other potential and the reference sheath position would be equally good, while the results obtained with the two methods obviously, coincide surprisingly well.

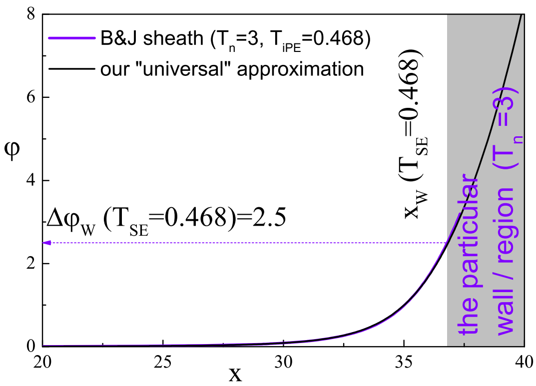

Figure 4: New sheath representation

Since no appreciable attempts towards formulating a consistent sheath theory with warm ions have been reported so far, the present findings may be considered as a milestone towards developing a possible fully universal parametric sheath solution. From our present work it emerges that such a solution is not expected to depend strongly on the ion temperature, provided that its shape is consistent with the adjacent quasi-neutral plasma. Artificial VDFs, however, including the "traditional" waterbag and *delta* functions, have been shown here to be bad candidates for searching such a universal solution.

References

[1] L. Kos, N. Jelić, S. Kuhn, and D. D. Tskhakaya, Phys. Plasmas **25**, 043509 (2018).
[2] R. C. Bissell and P. C. Johnson, Phys. Fluids **30**, 779 (1987).
[3] E. R. Harrison and W. B. Thompson, Proc. Phys. Soc. **74**, 145 (1959).
[4] L. Kos, N. Jelić, T. Gyergyek, S. Kuhn, and D. D. Tskhakaya, AIP Advances **8**, 105311 (2018).
[5] R. C. Davidson, *Methods in Nonlinear Plasma Theory* (Academic Press, New York and London, 1972).