# Numerical code for calculating plasma waves dispersion in relativistic magnetized plasma

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Propagation of electromagnetic waves in a plasma and a problem of plasma instabilities play a crucial role for both astrophysics and laboratory tasks. In order to determine which modes of plasma oscillations can exist and which of them can be unstable in the considered system it is required to solve dispersion equation:

$$F(\omega,\mathbf{k}) = \mid k_\alpha k_\beta - k^2\delta_{\alpha\beta} + \omega^2/c^2\varepsilon_{\alpha\beta} \mid = 0, \tag{1}$$

where $\mathbf{k} = \left(k_\perp, 0, k_\parallel\right)$ is a wave number with components along ($\|$) and transverse ($\perp$) to an external magnetic field, $\omega$ is a wave frequency, $\varepsilon_{\alpha\beta}(\omega,\mathbf{k}) = \delta_{\alpha\beta} + \sum_\sigma \varepsilon^{(\sigma)}_{\alpha\beta}$ – a dielectric tensor of the entire system, and $\varepsilon^{(\sigma)}_{\alpha\beta}$ presents the contribution of each plasma components (ions, electrons, particle beams etc). The tensor $\varepsilon^{(\sigma)}_{\alpha\beta}$ can be calculated in the the frameworks of different approaches such as cold plasma or MHD, with or without an external magnetic field. In practice, the considered systems have a significant temperature spread, a non-Maxwellian distribution and are immersed in an external magnetic field. Consequently, for correct description to be carried out it is necessary to investigate the full spectrum of oscillations using exact relativistic kinetic theory taking into account effects of a finite magnetic field.

In the classic kinetic approach, components of the tensor $\varepsilon_{\alpha\beta}$ for homogeneous in space plasma are expressed in terms of slowly converging sum of Bessel functions products under double integration:

$$\varepsilon^{(\sigma)}_{xx} = \frac{4\pi e^2}{\omega}\int d\mathbf{p} \sum_{n=-\infty}^{\infty} A_n \frac{n^2\Omega^2}{k_\perp} J_n^2; \tag{2}$$

where $\mathbf{p} = \left(p_\perp, 0, p_\parallel\right)$ is a particle momentum, $\Omega$ – a cyclotron frequency and $A_n$ – a function, which depends on derivatives of a particle momentum distribution. In this case, the calculation of $\varepsilon_{\alpha\beta}(\omega,\mathbf{k})$ is associated with significant difficulties. Therefore, researchers usually restrict consideration to either the case of a cold plasma or other simplifications and limiting cases. However, results of calculations in various models differ dramatically.

Basing on the symmetry of particle trajectories in a magnetic field we have suggested an approach [1] which allows us to use the finite integration instead of the infinite sums:

$$\sum_{n=-\infty}^{\infty} \frac{J_n^2(z)}{a-n} \longrightarrow \int_0^{2\pi} e^{-ia\varphi} J_0\left(2z\sin\frac{\varphi}{2}\right) d\phi, \quad (3)$$

where $a = \gamma\omega - k_{\parallel}p_{\parallel}/\Omega$, $z = k_{\perp}p_{\perp}/\Omega$, $\gamma = \sqrt{1+p_{\perp}^2+p_{\parallel}^2}$. Thus, for the tensor $\varepsilon_{\alpha\beta}(\omega,\mathbf{k})$ to be obtained it is necessary to calculate six independent components each of them is a triple integral: two integrations over the momentum space (in axial symmetry along a magnetic field direction) and a finite integral in the limits from 0 to $2\pi$ of the Bessel function of zero-order and a complex argument.

In such a formulation of the problem, computation of a separate component of the dielectric tensor is not computationally difficult. However, the need to study the full spectrum and the presence of many different oscillation branches in the magnetized plasma, which may be unstable, can make the problem rather cumbersome. To solve this problem efficiently, we has developed multithreaded computing complex in C++ language. This code is designed as a shared library, which makes possible to integrate it into third-party programs. The computing architecture can be either Nvidia GPGPU (via CUDA) or conventional multi-core CPU (via OpenMP). An interface to python implemented in cython language are also included in the complex. Basic functionality is connected with the dielectric tensor calculation. In addition, functions computing the function $F(\omega,\mathbf{k})$ value and its derivations $\partial F(\omega,\mathbf{k})/\partial\omega$ are implemented what is needed to find the roots of the dispersion equation. The code supports arbitrary particle distribution functions of the form $f\left(p_{\perp},p_{\parallel}\right)$. At the moment, it is possible to use Maxwell's anisotropic distribution ( fig. 1d):

$$f_M^{(\sigma)}\left(p_{\perp},p_{\parallel}\right) \propto \exp\left(-\frac{\left(p_{\perp}-P_{\perp}^{(\sigma)}\right)^2}{\Delta p_{\perp}^{(\sigma)2}} - \frac{\left(p_{\parallel}-P_{\parallel}^{(\sigma)}\right)^2}{\Delta p_{\parallel}^{(\sigma)2}}\right), \quad (4)$$

as well as the distribution function of beam particles which has an initially isotropic Maxwellian energy distribution and pitch-angle distribution:

$$f_P(\theta) \propto \exp\left(-\theta^2/\Delta\theta^2\right). \quad (5)$$

It is assumed that the electron beam is created in the magnetic field $B_0$ and then is transported to the higher magnetic field $B_1 = K\cdot B_0$ ($K > 1$). During this process the angular spread of the particles is significantly increasing (fig. 1e).

As an example, shown in Figure 1 are maps of the instability increment of the beam with the directional speed $v_b = 0.95c$ and $n_b = 0.005n_0$, where $c$ is the speed of light, $n_0$ – homogeneous plasma density.

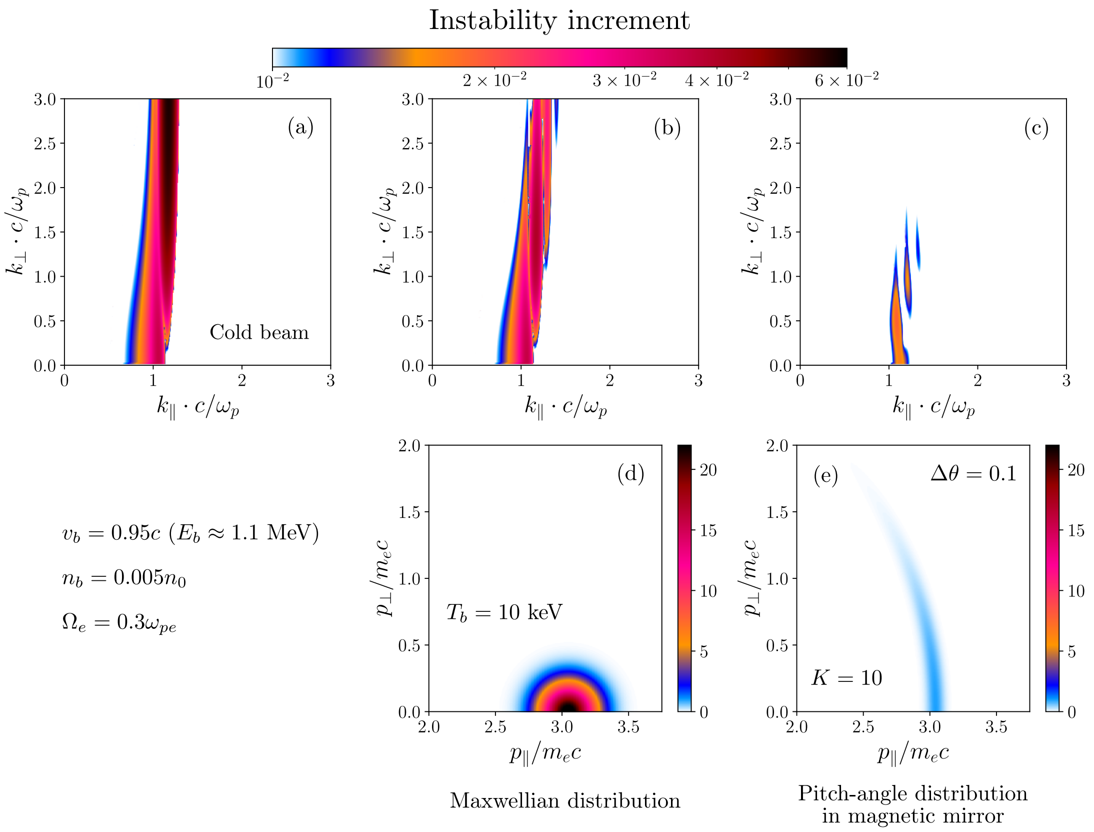


Figure 1: *(a-c) Growth rate map $\Gamma(k_\perp, k_\parallel)$ for the beam-plasma instability: (a) Cold beam; (b) Beam with Maxwellian distribution presented in (d); (c) Beam with pitch-angle distribution presented in (e).*

Plasma electrons are assumed as to be cold and ions – immobile. The entire system is immersed in a magnetic field characterized by the electron cyclotron frequency $\Omega_e = 0.3$. Hereinafter, all frequencies are in units of the plasma frequency $\omega_p = \sqrt{4\pi n_0 e^2/m_e}$ and wave vectors are in $\omega_p/c$. In Figure 1a, the beam is assumed to be cold; in Figure 1(b), the beam has isotropic Maxwell's distribution with $T_b = 10$ keV; in Figure 1(c) – pitch-angle distribution compressed in the magnetic mirror with $K = 10$. The grid step of wave vectors is $\Delta k = 0.01$. The search for solutions at each point was carried out using a neighboring point as an initial approximation. The search starts with the predictions of the cold theory ($k_\perp$=0.01, $k_\parallel$=$1/v_b$).

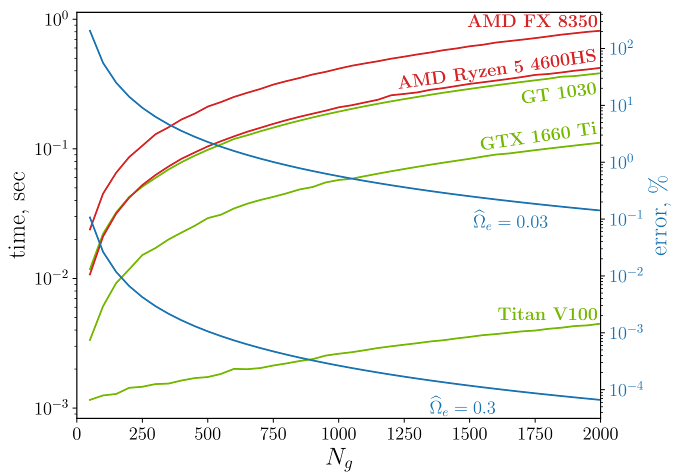


Figure 2: *The dependence of the computing time and accuracy on the number of segments in G-integrals.*

As one can see in Figure 1, there is a significant difference between the results obtained both in absolute value and in the localization of the most unstable modes.

Calculations of the integrals are implemented using rectangular integration. A quite small number of integration segments by momenta ($N_p = 50\dots150$) is usually sufficient to achieve acceptable accuracy and this number does not depend on point of space $(\omega, \mathbf{k})$, in which the tensor $\varepsilon_{\alpha\beta}$ is calculated. The situation is more complicated for calculating so-called G-integrals 3. The integrand is an oscillating function, the frequency of which increases with the growth of $k_\perp$ and decrease of a magnetic field. This leads to the need to increase the number of segments $N_g$ in order to keep the computation accuracy. The dependence of the computation accuracy of a separate component $\varepsilon_{\alpha\beta}$ on $N_g$ at $N_p = 150$ for Maxwellian distribution presented in 1d and an oblique oscillation mode is shown in Figure 2. One can see that the decrease of a magnetic field significantly affects the computation accurate (blue curves). Therefore, it is necessary to carefully monitor the accuracy of these integrals. The computational time for various devices are also presented in Figure 2. Time required to compute all 25806 points of the increment map shown in Figure 1b in the dependence on equipment is in Table 1. Graphics coprocessors proved to be the most efficient hardware tested.

Table 1: *Calculation time.*

| Processor | FP64 cores (freq., GHz) | Time, minutes |
|---|---|---|
| Nvidia Titan V100 | 2560 (1.23) | 6 |
| Nvidia GTX 1660 Ti | 48 (1.5) | 39 |
| Nvidia GTX 1650 | 28 (1.5) | 60 |
| Nvidia GT 1030 | 12 (1.2) | 114 |
| AMD Ryzen 7 3800X | 8 (3.9) | 98 |
| AMD Ryzen 5 4600HS | 6 (3) | 143 |
| Intel Xeon E5-2630 v3 | 8 (2.4) | 211 |
| AMD FX 8350 | 8 (4) | 310 |

A recent example of the use of this algorithm is finding a regime in which the most unstable beam-driven modes from two counterpropagating electron beams satisfy the condition of three-wave interaction with an electromagnetic wave [2]. Using particle-in-cell simulations in a realistic model with continuous injection of electron beams into a finite-size plasma we have confirmed high power-to-emission conversion efficiency in this regime.

This work was supported by the grant MK-2676.2021.1.2 of the President of the Russian Federation.

Source files of this library: https://bitbucket.org/vvannenkov/displib

## References

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