Self-generated magnetic fields modelling within high-order Lagrangian magneto-hydrodynamics

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Abstract

The mechanism of magnetic field self-generation due to crossed gradients of density and temperature is known for a long time from the astrophysical context, but it finds its importance even within modelling of laser–target interaction and inertial confinement fusion in particular. Therefore, it is highly desirable to incorporate this phenomenon to the magneto-hydrodynamic description commonly used for magnetized dense plasmas. Specifically, Lagrangian framework modelling the physics of interaction in the moving fluid frame is considered. However, the classical approaches suffer from the detrimental self-amplification process known as the Biermann catastrophe. We propose a stable method for modelling of the Biermann battery effect within the two-temperature high-order curvilinear finite element hydrodynamics, which also maintain the magnetic field divergence-free. Construction of the method is reviewed and verified on a physically relevant simulation.

Introduction

The process of spontaneous generation of magnetic fields due to the crossed gradient of density and temperature was first investigated in the astrophysical context[1], but its importance for the laser plasma was recognized soon after[2]. Macroscopically, the misaligned gradients of pressure and density generate a solenoidal electric field to restore quasi-neutrality of the plasma, where this field induces the magnetic field in turn. However, also shocks in non-ideal magneto-hydrodynamics contribute to this process. This phenomenon holds a great importance for inertial confinement fusion (ICF)[3] and proto-galactic seeds[4] for example. Unfortunately, efforts in numerical modelling struggled to

correctly treat the inherently non-linear source term, where the discontinuities at the shock fronts led to the artificial self-amplification process known as the Biermann catastrophe [5]. Here, we extend the high-order curvilinear Lagrangian magneto-hydrodynamics[6] by the model of the Biermann battery, where different numerical schemes are compared and tested on the problem of an elliptical shock.

Numerical model

The Biermann battery effect is modelled through an additional source term in the equation of electric field. This approach guarantees that the induced magnetic field remains divergence-free even for the high-order finite elements used here[6]. Different numerical schemes are constructed to optimize performance for the finite element method, though the physical model remains identical.

The first considered model we denote as *naive*, since it directly discretizes the source term:

$$\vec{E}_B = -\frac{\nabla p_e}{e n_e}, \tag{1}$$

where p_e is the electron pressure, e elementary charge and n_e electron density. Unfortunately, pressure is discontinuous across a shock front, so discretization problems can be expected.

The second model assumes an ideal gas equation of state and drops the gradient parts of the source term, where only the term with the gradient of temperature is retained[5]:

$$\vec{E}_B = -\frac{1}{e n_e}\nabla(n_e k_B T_e) = -\frac{k_B T_e}{e}\nabla \ln n_e - \cancel{\frac{k_B}{e}\nabla T_e} = -\cancel{\frac{k_B}{e}\nabla(T_e \ln n_e)} + \frac{k_B \ln n_e}{e}\nabla T_e, \tag{2}$$

where the symbol k_B represents the Boltzmann constant. The advantage of this method is that the temperature is continuous across the shock front in presence of a thermal precursor. Hence the gradient is transferred from the density to temperature, the method is called *dual*.

The third model presented is named *dual pressure*, pointing to the fact that rather pressure than temperature is present in the coefficient:

$$\vec{E}_B = -\frac{k_B T_e}{e p_e}\nabla p_e = -\cancel{\frac{k_B}{e}\nabla(T_e \ln p_e)} + \frac{k_B \ln p_e}{e}\nabla T_e = \frac{k_B \ln n_e}{e}\nabla T_e + \frac{k_B}{e}\nabla(T_e(\ln T_e - 1)). \tag{3}$$

The manipulations then reveal that methods (2) and (3) are equivalent in essence, as long as only the physical model is considered.

The aforementioned method of finite elements is used for implementation of the models, but the dominant gradient must be suppressed computationally to prevent numerical deterioration of the results. This is performed by means of the Helmholtz decomposition. On a Lipschitz domain Ω, $H^1(\Omega)$-conforming finite element space $\mathscr{A}$, $H_{curl}(\Omega)$-conforming space $\mathscr{E}$ and $L_2(\Omega)$-conforming space $\mathscr{T}$ are constructed. Considering only the *dual pressure model* given by (3) in 2D for simplicity, the solved system for $T_e \in \mathscr{T}, A \in \mathscr{A}, \vec{E}_B \in \mathscr{E}$

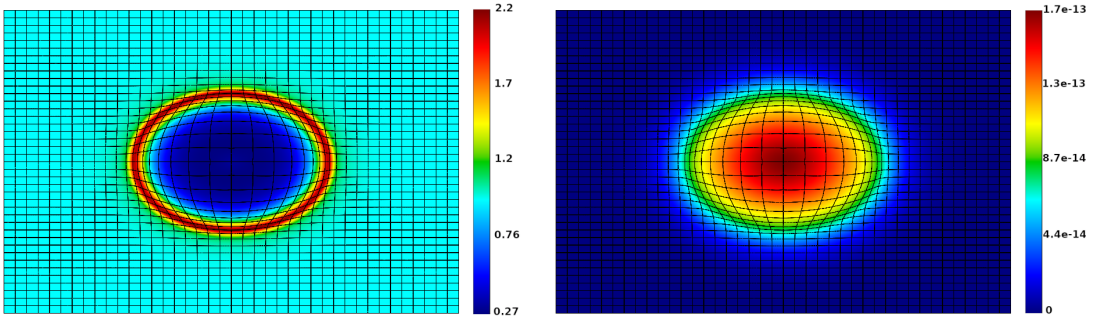
takes the form for $\forall \Lambda \in \mathscr{A}, \forall \vec{\xi} \in \mathscr{E}$ (E is the set of the elements in Ω and K the edges):

$$\int_\Omega \nabla A \cdot \nabla \Lambda \, \mathrm{d}V = \sum_E \int_E \frac{k_B \ln p_e}{e} \nabla T_e \cdot \nabla \times \Lambda \, \mathrm{d}V + \sum_K \int_K \{\frac{k_B \ln p_e}{e}\} [T_e] \nabla \times \Lambda \cdot \mathrm{d}\vec{S}, \tag{4}$$

$$\int_\Omega \vec{E}_B \cdot \vec{\xi} \, \mathrm{d}V = \int_\Omega A \nabla \times \vec{\xi} \, \mathrm{d}V. \tag{5}$$

Simulations

In order to test the proposed models of the Biermann battery, the problem of an elliptical shock wave is simulated[5]. The classical circular blast wave is obtained from the problem of the Sedov explosion with and the energy $e_0 = 1\,\mathrm{erg}$ deposited at the center of the domain $\Omega = (-2,+2)^2$ cm. The ambient medium has a negligible temperature, initial density $1\,\mathrm{g/cm^3}$ and the ideal gas equation of state for a fully ionized mono-atomic hydrogen. After the time $0.4\,\mathrm{s}$, the simulation is restarted with the profiles horizontally prolonged by 50 %. The heat diffusivity is set to $\approx 0.70\,\mathrm{cm^2/s}$ and the magnetic diffusivity equals $\approx 0.72\,\mathrm{cm^2/s}$. The two-temperature model was used[7], where the temperature relaxation coefficient was set to $\tau_{ei} = 5 \times 10^{-2}\,\mathrm{s}$.



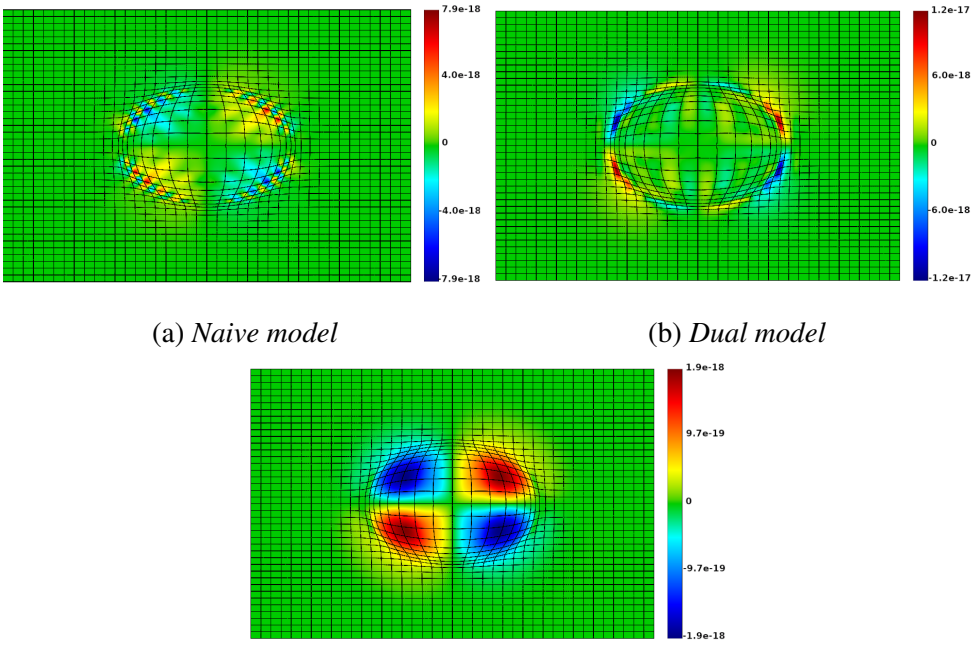
(a) *Mass density* $[g/cm^3]$ (b) *Electron temperature* $[eV]$

Figure 1: *Simulation results for the elliptical shock wave at time* $t = 0.6\,\mathrm{s}$.

The density and temperature profiles for 60×60 quadratic $\mathscr{T}$ and $\mathscr{A}$ and cubic $\mathscr{E}$ elements are plotted in Figure 1. The results obtained with the three models presented in the previous section are compared in Figure 2. The *naive model* exhibits strong oscillations at the shock in each element without any sign of convergence. The *dual model* achieves better performance, but weaker oscillations are still present in the downstream. The best results are obtained from the *dual pressure model*, giving smooth and convergent profiles.

Conclusions

The spontaneous magnetic fields arising from the Biermann battery process appear in important applications like ICF[3] or astrophysics[4, 1], but their modelling is numerically challenging at the shock fronts especially, where the naive model fails completely. Three different numerical models are constructed for high-order curvilinear finite element



(a) *Naive model* (b) *Dual model*

(c) *Dual pressure model*

Figure 2: *Magnetic field amplitudes [statT] generated by the Biermann battery models for the elliptical shock wave at time* $t = 0.6\,\mathrm{s}$. *See the accompanying text for description.*

codes[6, 7] and a convergent one is identified in the problem of an elliptical blast wave. The 3D extension and optimization of the method remain topics of the future research.

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