Energy and momentum conserving collisional bracket for the guiding-center Vlasov-Maxwell model

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Introduction

When dealing with the description of a plasma through the Vlasov-Maxwell-Landau system, both gyrokinetic and guiding-center theory are often applied to investigate solely the Vlasov-Maxwell part and the collision operator is either neglected or heavily approximated. This is in part because the dissipationless dynamics may be addressed with systematic reduction techniques via the Lie-transform perturbation theory (see e.g. [1]) but the dissipative part of the problem tends to lead to difficult truncation problems in trying to ensure that the effects of collisions do not violate, e.g., the laws of thermodynamics. Nevertheless, the modern formulation of collisional electrostatic gyrokinetics exhibits a metriplectic structure [2], an extension of the Poisson bracket formalism of classical mechanics to dissipative systems that obey the laws of thermodynamics (see e.g. [3, 4]). This observation prompts the question of whether some as-of-yet undiscovered metriplectic perturbation theory exists. The current work provides another indication of the possible existence of such theory and sheds further light into the issues of developing a collision operator for electromagnetic reduced plasma theories.

Metric bracket for collisions

The Landau operator, describing the effects due to small-angle Coulomb collisions between the species s and $\bar{s}$, can be expressed as

$$C_{s\bar{s}}(f_s, f_{\bar{s}}) = -\sum_{\bar{s}} \frac{\nu_{s\bar{s}}}{m_s} \frac{\partial}{\partial \boldsymbol{v}} \cdot \int \delta(\boldsymbol{x}-\bar{\boldsymbol{x}}) f_s(\boldsymbol{z}) f_{\bar{s}}(\bar{\boldsymbol{z}}) \mathbb{Q}(\boldsymbol{v}-\bar{\boldsymbol{v}}) \cdot \boldsymbol{\Gamma}_{s\bar{s}}(\mathscr{S}, \boldsymbol{z}, \bar{\boldsymbol{z}}) d\bar{\boldsymbol{z}}. \tag{1}$$

The coordinates $\boldsymbol{z} = (\boldsymbol{x}, \boldsymbol{v})$ and $\bar{\boldsymbol{z}} = (\bar{\boldsymbol{x}}, \bar{\boldsymbol{v}})$ refer to different phase-space locations, the vector $\boldsymbol{\Gamma}_{s\bar{s}}(\mathscr{A}, \boldsymbol{z}, \bar{\boldsymbol{z}}) = \frac{1}{m_s}\frac{\partial}{\partial \boldsymbol{v}}\frac{\delta \mathscr{A}}{\delta f_s}(\boldsymbol{z}) - \frac{1}{m_{\bar{s}}}\frac{\partial}{\partial \bar{\boldsymbol{v}}}\frac{\delta \mathscr{A}}{\delta f_{\bar{s}}}(\bar{\boldsymbol{z}})$ and $\mathbb{Q}(\boldsymbol{\xi}) = \left(\mathbb{I} - \boldsymbol{\xi}\boldsymbol{\xi}/|\boldsymbol{\xi}|^2\right) 1/|\boldsymbol{\xi}|$ is the familiar scaled projection matrix. We can express the weak form of (1) as

$$\sum_s \int g_s(\boldsymbol{z}) C_{s\bar{s}}(f_s, f_{\bar{s}}) d\boldsymbol{z} = \sum_{s\bar{s}} \frac{1}{2} \iint \boldsymbol{\Gamma}_{s\bar{s}}(\mathscr{G}, \boldsymbol{z}, \bar{\boldsymbol{z}}) \cdot \mathbb{W}_{s\bar{s}}(\boldsymbol{z}, \bar{\boldsymbol{z}}) \cdot \boldsymbol{\Gamma}_{s\bar{s}}(\mathscr{S}, \boldsymbol{z}, \bar{\boldsymbol{z}}) d\bar{\boldsymbol{z}} d\boldsymbol{z}, \tag{2}$$

where $\mathscr{G} = \int g_s(\boldsymbol{z}) f_s(\boldsymbol{z}) d\boldsymbol{z}$ is a functional and $\mathbb{W}_{s\bar{s}}(\boldsymbol{z}, \bar{\boldsymbol{z}}) = \nu_{s\bar{s}} \delta(\boldsymbol{x}-\bar{\boldsymbol{x}}) f_s(\boldsymbol{z}) f_{\bar{s}}(\bar{\boldsymbol{z}}) \mathbb{Q}(\boldsymbol{v}-\bar{\boldsymbol{v}})$ is a positive semi-definite matrix. One could view $g_s(\boldsymbol{z})$ as a test function.

This rather peculiar form enables a straightforward identification of a functional bracket

$$(\mathscr{A},\mathscr{B}) = \sum_{s,\bar{s}} \frac{1}{2} \iint \boldsymbol{\Gamma}_{s\bar{s}}(\mathscr{A},\boldsymbol{z},\bar{\boldsymbol{z}}) \cdot \mathbb{W}_{s\bar{s}}(\boldsymbol{z},\bar{\boldsymbol{z}}) \cdot \boldsymbol{\Gamma}_{s\bar{s}}(\mathscr{B},\boldsymbol{z},\bar{\boldsymbol{z}}) d\bar{\boldsymbol{z}} d\boldsymbol{z}, \tag{3}$$

in terms of which collisional evolution of arbitrary functionals can be generalized to the functional differential equation $\frac{d\mathscr{A}}{dt}\Big|_{\text{coll}} = (\mathscr{A},\mathscr{S})$. The bracket (3) has the kinetic energy $\mathscr{K} = \sum_s \int f_s(\boldsymbol{z},t) m_s |\boldsymbol{v}|^2/2 d\boldsymbol{z}$, the momentum $\mathscr{P} = \sum_s \int f_s(\boldsymbol{z},t) m_s \boldsymbol{v} d\boldsymbol{z}$, and the mass $\mathscr{M}_s = \int m_s f_s(\boldsymbol{z},t) d\boldsymbol{z}$ functionals as invariants as a result of the conditions

$$\boldsymbol{\Gamma}_{s\bar{s}}(\mathscr{M}_s,\boldsymbol{z},\bar{\boldsymbol{z}}) = 0, \quad \delta(\boldsymbol{x}-\bar{\boldsymbol{x}}) \boldsymbol{\Gamma}_{s\bar{s}}(\mathscr{P},\boldsymbol{z},\bar{\boldsymbol{z}}) = 0, \quad \boldsymbol{\Gamma}_{s\bar{s}}(\mathscr{K},\boldsymbol{z},\bar{\boldsymbol{z}}) \cdot \mathbb{W}_{s\bar{s}}(\boldsymbol{z},\bar{\boldsymbol{z}}) = 0. \tag{4}$$

The convenience of the bracket formulation is that it brings the collisional evolution on equal footing with the infinite-dimensional Hamiltonian formulation of the dissipationless Vlasov-Maxwell part [5] and formulates the kinetic system as whole in terms of the so-called metriplectic dynamics of arbitrary functionals [3, 4].

Collisional bracket for the guiding-center Vlasov-Maxwell system

The guiding-center Vlasov-Maxwell system has a variational structure and conserved quantities that can be identified via analysis of the system's Noether symmetries [6]. The global invariants are the total energy and momentum functionals

$$\mathscr{H}^{\text{gc}}[F,\boldsymbol{E},\boldsymbol{B}] = \sum_s \int K_s F_s d\boldsymbol{Z}_s^{\text{gc}} + \frac{1}{8\pi} \int \left(|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2 \right) d\boldsymbol{x}, \tag{5}$$

$$\mathscr{P}^{\text{gc}}[F,\boldsymbol{E},\boldsymbol{B}] = \sum_s \int m_s v_\parallel \boldsymbol{b} F_s d\boldsymbol{Z}_s^{\text{gc}} + \frac{1}{4\pi c} \int \boldsymbol{E} \times \boldsymbol{B} d\boldsymbol{x}, \tag{6}$$

with $K = \frac{1}{2} m v_\parallel^2 + \mu B$ being the individual guiding-center kinetic energy. The perturbation theory compatible with preserving Hamiltonian structures primarily operates at the level of the Lagrangian and not the Poisson structure. While this arrangement guarantees that truncations introduced to the perturbed Lagrangian facilitate a Poisson structure that satisfies the Jacobi identity, it does not instruct us on how to transform general brackets and functional derivatives: the truncation problem in applying the Lie-transformation perturbation theory to the Poisson structure still persists [7]. The alternative way to an appropriate collisional bracket is to look for a structure similar to (3) and to appropriately modify parts of it while simultaneously juggling with the conserved quantities. This results in a bracket

$$(\mathscr{A},\mathscr{B})^{\text{gc}} = \sum_{s,\bar{s}} \frac{1}{2} \iint \boldsymbol{\Gamma}^{\text{gc}}_{s\bar{s}}(\mathscr{A},\boldsymbol{Z},\bar{\boldsymbol{Z}}) \cdot \mathbb{W}^{\text{gc}}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}}) \cdot \boldsymbol{\Gamma}^{\text{gc}}_{s\bar{s}}(\mathscr{B},\boldsymbol{Z},\bar{\boldsymbol{Z}}) d\bar{\boldsymbol{Z}}^{\text{gc}}_{\bar{s}} d\boldsymbol{Z}^{\text{gc}}_s, \tag{7}$$

where $\mathbb{W}^{\text{gc}}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}}) = \nu_{s\bar{s}} \delta(\boldsymbol{X}-\bar{\boldsymbol{X}}) F_s(\boldsymbol{Z}) F_{\bar{s}}(\bar{\boldsymbol{Z}}) \mathbb{Q}(\boldsymbol{\Gamma}^{\text{gc}}_{s\bar{s}}(\mathscr{H}^{\text{gc}},\boldsymbol{Z},\bar{\boldsymbol{Z}}))$ and the vector $\boldsymbol{\Gamma}^{\text{gc}}_{s\bar{s}}(\mathscr{A},\boldsymbol{Z},\bar{\boldsymbol{Z}})$ is

$$\boldsymbol{\Gamma}^{\text{gc}}_{s\bar{s}}(\mathscr{A},\boldsymbol{Z},\bar{\boldsymbol{Z}}) = \left(\frac{\boldsymbol{b}}{m} \frac{\partial}{\partial v_\parallel} + \frac{\Omega \boldsymbol{b} \times \boldsymbol{\rho}_0}{B} \frac{\partial}{\partial \mu} \right) \frac{\delta \mathscr{A}}{\delta F}\Big|_{s,\boldsymbol{Z}} - \left(\frac{\boldsymbol{b}}{m} \frac{\partial}{\partial v_\parallel} + \frac{\Omega \boldsymbol{b} \times \boldsymbol{\rho}_0}{B} \frac{\partial}{\partial \mu} \right) \frac{\delta \mathscr{A}}{\delta F}\Big|_{\bar{s},\bar{\boldsymbol{Z}}}, \tag{8}$$

with Ω the cyclotron frequency and $\boldsymbol{\rho}_0$ the lowest order expression for the gyroradius. The first part in (8) is to be evaluated at the position $\boldsymbol{Z}$ with respect to the species s parameters and the second part in a similar manner but at $\bar{\boldsymbol{Z}}$ and with respect to species $\bar{s}$.

The bracket has the total energy (5) and momentum (6) as invariants. As $\boldsymbol{\Gamma}^{\mathrm{gc}}_{s\bar{s}}(\mathscr{H}^{\mathrm{gc}},\boldsymbol{Z},\bar{\boldsymbol{Z}})$ belongs to the null space of $\mathbb{W}^{\mathrm{gc}}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}})$ it ensures $(\mathscr{H}^{\mathrm{gc}},\mathscr{A})^{\mathrm{gc}}=0$ with respect to arbitrary $\mathscr{A}$. Further, the presence of $\delta(\boldsymbol{X}-\bar{\boldsymbol{X}})$ in $\mathbb{W}^{\mathrm{gc}}_{s\bar{s}}$ ensures that $(\mathscr{P}^{\mathrm{gc}},\mathscr{A})=0$. The choices of (8) and $\mathbb{W}^{\mathrm{gc}}_{s\bar{s}}(\boldsymbol{Z},\bar{\boldsymbol{Z}})$ together with the general expression for the bracket (7) now guarantee that if the collisional evolution of functionals were to be given by $\left.\frac{d\mathscr{A}}{dt}\right|_{\mathrm{coll}}=(\mathscr{A},\mathscr{S})^{\mathrm{gc}}$ and driven by the entropy, both the energy and momentum conservation would be satisfied and the entropy dissipation would be guaranteed. While the structure of the collisional bracket (7) might appear somewhat intimidating, parts of it can be handled analytically and the gyroangle dependency in the bracket averaged in terms of the complete elliptic integrals.

The gyrokinetic problem

The collisional bracket for the guiding-center Vlasov-Maxwell-Landau system that we have presented and analyzed is the first one of its kind for any temporally reduced electromagnetic kinetic plasma model. Nevertheless there exists a collisional bracket for the electrostatic gyrokinetic model [2]. This raises the question whether analogous brackets or energetically-consistent collision operators for other reduced electromagnetic kinetic plasma theories exist.

The particle velocity can be represented in the reduced coordinates to a reasonable accuracy as $\boldsymbol{v}=\dot{\boldsymbol{X}}+\dot{\boldsymbol{\rho}}_0$. Where $\dot{\boldsymbol{X}}$ captures the parallel streaming and the slower drifts while $\dot{\boldsymbol{\rho}}_0$ captures the fast Larmor rotation. Further, we know that the velocity-space derivative of a particle phase-space function f can be expressed in terms of the non-canonical particle phase-space Poisson bracket. In the electrostatic case [2], the logical expression for the vector $\boldsymbol{\Gamma}(\mathscr{A})$ in the collisional bracket is then

$$\boldsymbol{\Gamma}(\mathscr{A})=\left\{\boldsymbol{X}+\boldsymbol{\rho}_0,\frac{\delta\mathscr{A}}{\delta F}\right\}_z-\left\{\boldsymbol{X}+\boldsymbol{\rho}_0,\frac{\delta\mathscr{A}}{\delta F}\right\}_{\bar{z}}. \tag{9}$$

In perturbed electromagnetic theories, such as the drift-kinetic and gyrokinetic ones, things are different. The single drift-center or gyrocenter Lagrangian is presented as

$$L=\vartheta_\alpha\dot{z}^\alpha-K(\boldsymbol{E}_1,\boldsymbol{B}_1)+(e/c)A_{1,i}\dot{X}^i-e\varphi_1, \tag{10}$$

with ϑ_α the six time-independent components of the unperturbed guiding-center one-form, K the kinetic energy function, and $(\boldsymbol{A}_1,\varphi_1)$ and $(\boldsymbol{B}_1,\boldsymbol{E}_1)$ the perturbed electromagnetic potentials and fields respectively. In this case, the particle velocity in reduced coordinates becomes

$$\boldsymbol{v}=\{\boldsymbol{X}+\boldsymbol{\rho}_0,K\}-\{\boldsymbol{X}+\boldsymbol{\rho}_0,\boldsymbol{X}\}\cdot e\boldsymbol{E}_1. \tag{11}$$

Following the drift-kinetic system documented in [8], where the canonical toroidal momentum and energy functionals are conserved, one finds that

$$\frac{\delta \mathscr{P}_\phi}{\delta F} \neq p_\phi, \qquad \left\{ \boldsymbol{X} + \boldsymbol{\rho}_0, \frac{\delta \mathscr{H}}{\delta F} \right\} \neq \boldsymbol{v}, \tag{12}$$

with p_ϕ the single guiding-center canonical toroidal momentum. There does not appear to be an immediate, simple way to modify (9) so that one would recover $\boldsymbol{\Gamma}(\mathscr{H}) = \boldsymbol{v} - \bar{\boldsymbol{v}}$, where $\boldsymbol{v}$ is given by (11), and $\boldsymbol{\Gamma}(\mathscr{P}_\phi) = \{\boldsymbol{X} + \boldsymbol{\rho}_0, p_\phi\}$ which guarantee the conservation properties for the electrostatic case. Based on our findings, we conclude that a systematic tool for performing asymptotic dynamical reduction of collisional process, or more precisely of metric brackets, is necessary. Although Lie-transform perturbation theory is an established tool to handle asymptotic dynamical reduction of dissipation-free dynamics, no similar compatible theory exists yet to handle structure-preserving dissipative dynamics.

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