Analytical calculation of the Orbital Spectrum of Guiding Center motion: Zero and Finite Drift Width effects and consequences in resonant transport

Y. Antonenas, G. Anastassiou, T. Bournelis, and Y. Kominis
National Technical University of Athens, Athens, Greece

Introduction

The Orbital Spectrum of the Guiding Center (GC) motion in an axisymmetric toroidal magnetic field determines the resonant effects of non-axisymmetric perturbations on particle, energy and momentum transport. Finite-Drift-Width effects modify the Orbital Frequencies and the respective resonance conditions. In this work we calculate analytical expressions for the bounce/transit frequencies as well as for the bounce/transit averaged toroidal precession and gyration frequencies and pinpoint the location of resonances with non-axisymmetric perturbation in the phase space of the GC motion.

Guiding Center Hamiltonian

A general axisymmetric toroidal magnetic configuration consisting of nested toroidal flux surfaces can be represented in White–Boozer [1] coordinates as

$$B = g(\psi)\nabla\zeta + I(\psi)\nabla\theta + \delta(\psi,\theta)\nabla\psi_p$$

where ζ and θ are the toroidal and poloidal angles. The toroidal flux ψ is related to the poloidal flux ψ_p through the safety factor $q(\psi) = d\psi/d\psi_p$. The functions g and I are related to the poloidal and toroidal currents and δ is related to the non-orthogonality of the coordinate system.

The GC motion of a charged particle is described by the Hamiltonian $H = \rho_{||}^2\boldsymbol{B}^2/2 + \mu\boldsymbol{B}$, where $\boldsymbol{B}$ is the magnetic field, μ is the magnetic moment and $\rho_{||}$ is the velocity component parallel to the magnetic field. The three couples of canonical conjugate variables for this GC Hamiltonian are (μ,ξ), (P_θ,θ) and (P_ζ,ζ) with $P_\theta = \psi + \rho_{||}I(\psi)$ and $P_\zeta = \rho_{||}g(\psi) - \psi_p$ [1]. The Hamiltonian can be written with respect to these canonical variables, as $H(P_\theta,\theta,P_\zeta,\zeta,\mu,\xi)$.

A general canonical transformation to drift orbit deviation variables transforms the above Hamiltonian to a new (barred) variable set [2]. The physical meaning of the new canonical variables becomes obvious for a Large Aspect Ratio (LAR) cylindrical equilibrium described by $g = 1, I = 0$, and $B = 1 - r\cos\theta$, where $r = \sqrt{2\psi}$ [1]. In this case the initial variables take the form $P_\theta = \psi$,

$P_\zeta = \rho_{||} - \psi_p(\psi)$ and the barred variables $\bar{P}_\theta = \psi - \psi_0$, $\bar{\theta} = \theta - q^{-1}(\psi)$, $\bar{P_\zeta} = P_\zeta + \psi_p(\psi)$, $\bar{\zeta} = \zeta$ [2].

The above transformation allows for the Full/Zero Drift Width (FDW/ZDW) formulation. According to the ZDW formulation, the magnetic field is evaluated on a particular magnetic surface of reference, $B(\psi,\theta) \to B(\psi_0,\theta)$, whereas the GC deviation from a field line and particle drifts are possible. The corresponding phase spaces are depicted in Fig. 1, also compared to the, widely used in the literature, pendulum-like Hamiltonian $\bar{H}'_{ZDW}$ [1].

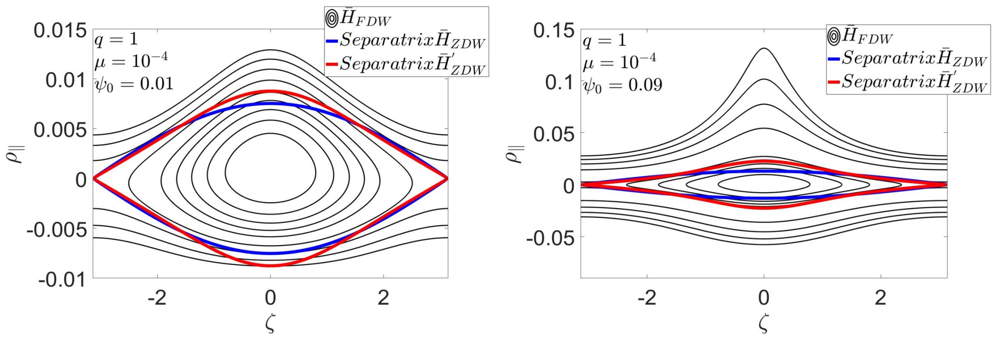

Figure 1: Phase space $(\rho_{||}, \zeta)$ of the Hamiltonian $\bar{H}_{FDW}$ for $q = 1$, $\mu = 10^{-4}$ and $\psi_0 = 0.01$ (left) and $\psi_0 = 0.09$ (right). The separatrices between bounce and transit motion according to $\bar{H}_{ZDW}$ and $\bar{H}'_{ZDW}$ are depicted by blue and red lines, respectively. Therefore, a particle that is described as being trapped according to $\bar{H}'_{ZDW}$ can be actually passing according to $\bar{H}_{ZDW}$ and vise versa. Both ZDW Hamiltonians describe GC orbits that are symmetric with respect to $\rho_{||}$, whereas according to $\bar{H}_{FDW}$ positive (co-passing) and negative (counter-passing) orbits are not symmetric. These differences strongly depend on the pitch angle and the flux surface of reference.

Analytical calculation of the Actions and the Orbital Frequencies

The action-angle transformation $\bar{H}(\bar{P_\zeta}, \bar{\zeta}, \bar{P}_\theta, \bar{\theta}, \bar{\mu}, \bar{\xi}) \leftrightarrow \hat{H}(J_\zeta, J_\theta, J_\xi)$ allows for the analytical calculation of the orbital frequencies for the three degrees of freedom. Therefor:

$$\widehat{\omega}_\zeta = \frac{\partial \hat{H}}{\partial J_\zeta}, \qquad \widehat{\omega}_\theta = -\widehat{\omega}_\zeta \frac{\partial \hat{H}}{\partial J_\theta}, \qquad \widehat{\omega}_\xi = -\widehat{\omega}_\zeta \frac{\partial \hat{H}}{\partial J_\xi}$$

where $\widehat{\omega}_\zeta$ is the bounce/transit frequency, $\widehat{\omega}_\theta$ is the bounce/transit-averaged toroidal precession frequency, $\widehat{\omega}_\xi$ is the bounce/transit-averaged gyration frequency and $(J_\zeta, J_\theta, J_\xi)$ are the actions. The three actions and the bounce (b) / transit (t) frequencies are given by the analytical expressions

$$J_\zeta^b = \frac{8q(\psi_0)\sqrt{\mu r}}{\pi\eta(1-r)}[(\eta k - 1)\Pi(\eta k, k) + K(k)], \; J_\zeta^t = \frac{4q(\psi_0)\sqrt{\mu r}}{\pi\eta(1-r)\sqrt{k}}[(\eta k - 1)\Pi(\eta, k^{-1}) + K(k^{-1})]$$

$$J_\theta^{b,t} = -q(\psi_0)\big(P_\zeta + \psi_p(\psi_0)\big) \qquad J_\xi^{b,t} = \mu$$

$$\widehat{\omega}_\zeta^b = \frac{\pi(1-r)\sqrt{\mu r}}{2q(\psi_0)\Pi(\eta k, k)} \qquad \widehat{\omega}_\zeta^t = \frac{\pi\sqrt{k}(1-r)\sqrt{\mu r}}{2q(\psi_0)\Pi(\eta, k^{-1})}$$

where $r = \sqrt{2\psi_0}$, $k = \frac{E-\mu(1-r)}{2\mu r}$, $n = -\frac{2r}{1-r}$. The analytically expressions for the frequencies $\widehat{\omega}_\theta$ and $\widehat{\omega}_\xi$ are too lengthy to be given here. Frequencies' dependence on the trapping parameter (k) is depicted in Fig. 2.

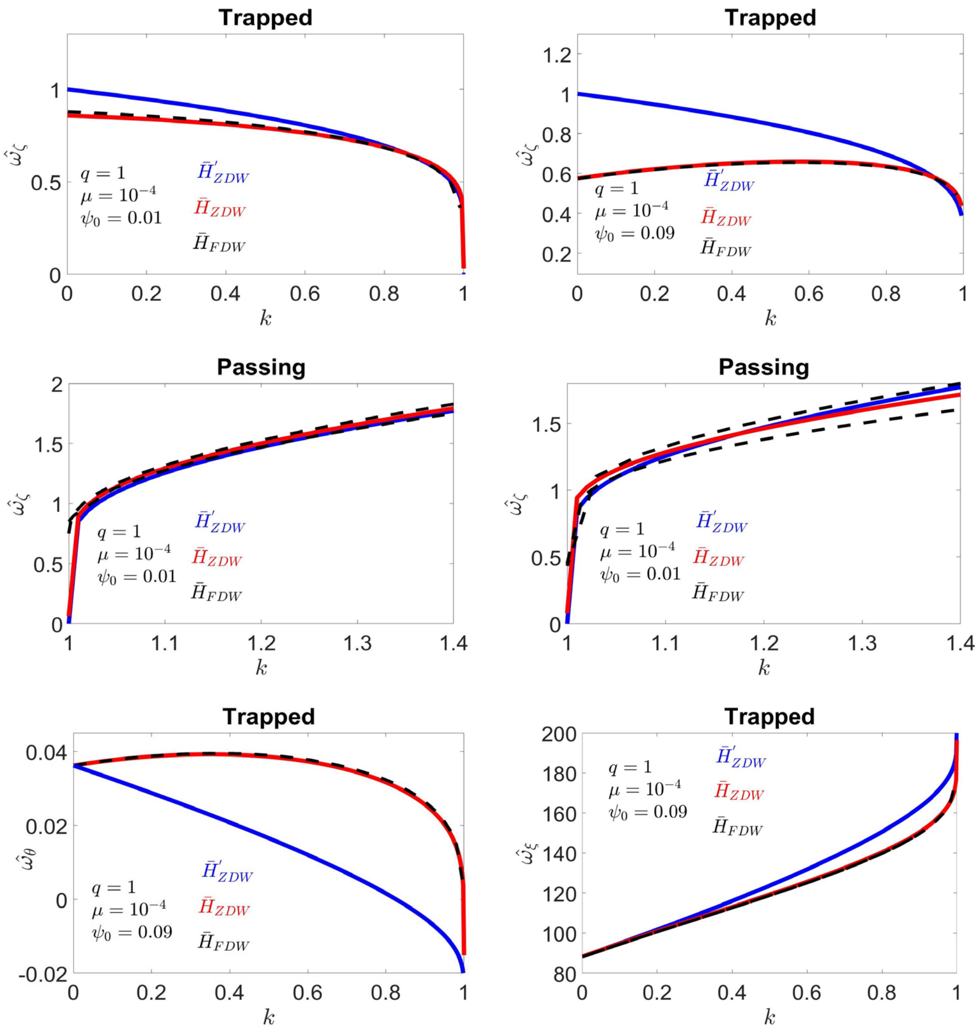

Figure 2: Analytically (solid lines) and numerically (dashed lines) calculated bounce (first row) and transit (second row) frequencies according to $\overline{H}'_{ZDW}$ (blue line), $\overline{H}_{ZDW}$ (red line) and $\overline{H}_{FDW}$ (dashed line). In contrast to the pendulum-like Hamiltonian case $\overline{H}'_{ZDW}$, the frequency is non-monotonic with respect to k (energy), in accordance with FDW numerical calculations, indicating the existence of two trapped orbits with different energy but equal frequency. The FDW Hamiltonian $\overline{H}_{FDW}$ describes asymmetric orbits and consequently different frequencies for co-passing and counter-passing orbits. The analytical formula for the ZDW Hamiltonian corresponds to an intermediate frequency with respect to the two branches of the numerically calculated FDW frequencies, whereas the analytical formula for the pendulum-like Hamiltonian $\overline{H}'_{ZDW}$ tends to follow one of the two branches. The third row depicts the corresponding bounce-averaged toroidal precession (left) and gyro (right) frequencies.

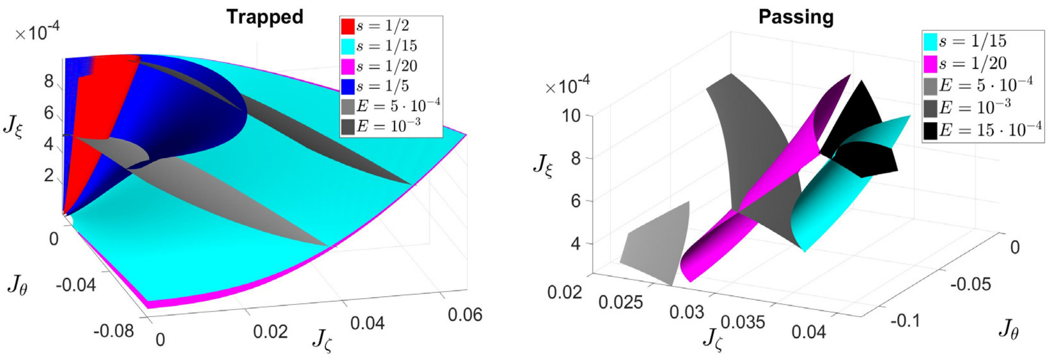

Figure 3: Resonance and Energy surfaces were analytically calculated according to the resonance condition and the energy conservation condition respectively. The interactions with non-axisymmetric perturbations take place in a constant energy surface. It is shown that resonance conditions for $s = 1/2$ and $s = 1/5$ are not met for the passing motion and particles with energy $E = 15 \cdot 10^{-4}$ are only passing.

Non-axisymmetric Perturbations and Resonance Conditions

The presence of non-axisymmetric perturbations results in a Hamiltonian of the form:

$$H = \hat{H}(J_\zeta, J_\theta, J_\xi) + \sum_{m,n,l} H_{m,n,l}\,(J_\zeta, J_\theta, J_\xi) \exp\left[i(m\hat{\theta} - n\hat{\zeta})\right]$$

The perturbations affect particle and momentum transport in a resonant fashion. The interactions with perturbations take place in a constant energy surface. The resonance condition $m\hat{\omega}_\theta - n\hat{\omega}_\zeta = 0$ and the energy conservation condition $\hat{H}(J_\zeta, J_\theta, J_\xi) = C$ allow to pinpoint the exact locations of resonances in the action space as shown in Fig. 3.

Summary and Conclusions

The ZDW formulation leads to analytical expressions for the frequencies, which significantly differ from those corresponding to the widely used pendulum-like Hamiltonian, and show a remarkable agreement with numerically calculated frequencies. The Action-Angle transformation allows for determining the resonance conditions under particle interaction with non-axisymmetric perturbations that affect energy, momentum and particle transport in toroidal plasma configurations and the application of standard canonical perturbation methods as well as the systematic dynamical reduction and the formulation of a bounce-kinetic description.

[1] R.B. White and M.S. Chance, *Hamiltonian guiding center drift orbit calculation for plasmas of arbitrary cross section*, Phys. Fluids 27, 2455–2467 (1984).

[2] Y. Antonenas, G. Anastassiou, and Y. Kominis, *Analytical Calculation of the Orbital Spectrum of the Guiding Center Motion in Axisymmetric Magnetic Fields,* J. Plasma Phys. 87, 855870101 (2021).

This work has been carried out within the framework of the EUROfusion Consortium and has received funding from the Euratom research and training programme 2014-2018 and 2019-2020 under grant agreement No 633053. The views and opinions expressed herein do not necessarily reflect those of the European Commission.