Hierarchical approach for energetic particle transport in 1-dimensional uniform plasmas

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Abstract The importance of the beam-plasma system (BPS) in fusion physics relies on its capability in reproducing relevant features of energetic particles interacting with the Alfvénic spectrum [1, 2]. We analyze here a multi-level hierarchy of the Vlasov-Poisson (VP) induced transport in order to characterize the underlying physical processes.

Hamiltonian description of the beam-plasma interaction The BPS faces the resonant dynamics of a fast particle beam injected into a 1D plasma, which is treated as a cold linear dielectric medium supporting electrostatic turbulence. We adopt the Hamiltonian formulation[1] of the problem [3] where the broad energetic particle beam self-consistently evolves in the presence of M linearly unstable modes, each one almost at the plasma frequency, i.e. $\omega \simeq \omega_p$:

$$\bar{x}_i' = u_i\,, \qquad u_i' = \sum_\ell \left(i\ell\,\bar{\phi}_\ell\, e^{i\ell\bar{x}_i} + c.c. \right)\,, \qquad \bar{\phi}_\ell' = -i\bar{\phi}_\ell + \frac{i\eta}{2\ell^2 N}\sum_\ell e^{-i\ell\bar{x}_i}\,. \tag{1}$$

The resonance conditions write $\ell u_r = \omega/\omega_p \simeq 1$ (u_r being the resonant velocities) and the warm beam is initialized with an assigned distribution function (DF) $\bar{F}_B(u)$, with $S = \int du \bar{F}_B(u)$.

Vlasov-Poisson system The BPS can be treated kinetically via the VP coupled system expressed using the Fourier components of the electric field ($E_k(t)$) and of the beam DF ($f_k(t,v)$):

$$\partial_t f_k = -ikv\, f_k + \frac{e}{m}\sum_{k'} E_{k'}\partial_v f_{k-k'}\,, \qquad \partial_t E_k = -i\omega_p E_k + \frac{2\pi e \omega_p}{k}\int_{-\infty}^{\infty} dv f_k\,. \tag{2}$$

Due to the initial spatial homogeneity of the system, $f_0 \equiv f_B(t,v)/L$ is the only k having non-zero initial conditions and it is governed by the following dimensionless transport equation:

$$\partial_\tau \bar{f}_B(\tau,u) = \partial_u \Gamma_{bps}(\tau,u) = \partial_u \left[4\pi \sum_\ell \left[\ell\bar{\phi}_\ell^b\, \bar{f}_\ell^a - \ell\bar{\phi}_\ell^a\, \bar{f}_\ell^b \right] \right]\,, \tag{3}$$

[1]Notation: The 1D cold plasma is taken as a periodic slab of length L. Beam particle positions and velocities are x_i and v_i, N is the total particle number. The electrostatic potential $\varphi(x,t)$ is expressed in terms of the Fourier components $\varphi_k(t)$ (k is the wave-number). Introducing the beam to plasma density ratio $\eta = n_B/n_p \ll 1$, we use the dimensionless variables: $\bar{x}_i = x_i(2\pi/L)$, $\tau = t\omega_p$, $u_i = \bar{x}_i' = v_i(2\pi/L)/\omega_p$, $\ell = k(2\pi/L)^{-1}$ (integers), $\phi_\ell = (2\pi/L)^2 e\varphi_k/m\omega_p^2$ and $\bar{\phi}_\ell = \phi_\ell e^{-i\tau}$. The prime denotes τ derivative.

where we used $u = 1/\ell$, $E_k = -ik\bar{\varphi}_k = k\bar{\varphi}_k^b - ik\bar{\varphi}_k^a$ (here, $\bar{\varphi}_k = \bar{\phi}_\ell(m\omega_p^2/(2\pi/L)^2 e))$ and $\bar{f}(\tau, u, \bar{x})$ is taken from the histogram of the phase-space N-body simulations. Eq.(3) corresponds to the zeroth level of the hierarchy scheme we are analyzing: it allows to define the proper form of the fluxes Γ_{bps}, evaluated by sampling $\bar{\phi}_\ell(\tau)$ and $\bar{f}_\ell(\tau, u)$ from simulations of Eq.(1), to be compared with the other approximation levels defined in what follows. The evolution of the DF matches exactly the profiles obtained from the fully self-consistent scheme.

Diagonal reduced Vlasov-Poisson system The single function f_k is assumed to receive mainly contribution from the correspondent harmonics ($k' = k$ in Eq.(2)). It thus satisfies:

$$\partial_t f_k = -ikv f_k + \frac{e}{m} E_k \partial_v f_0 \,, \qquad \Rightarrow \qquad f_k(t,v) = \frac{e}{m}\int_0^t dt' E_k(t') e^{ikv(t'-t)} \partial_v f_0(t',v) \,,$$

obtaining a diagonal reduced transport equation for f_0 (here and in the following, we use $k > 0$):

$$\partial_t f_0(t,v) - \frac{e^2}{m^2}\sum_k \Big[E_k\, \partial_v\Big(\int_0^t dt' E_k^*(t') e^{ikv(t'-t)} \partial_v f_0(t',v)\Big) + c.c.\Big] = 0 \,. \qquad (4)$$

The electric field can be set as $E_k(t') = E_k(t)\, \exp\big[-i\int_t^{t'} dt'' \omega_k(t'')\big]$ getting, form Eq.(4) and Eq.(2) (right), a Dyson-like system for the evolution of f_0 and of the spectrum:

$$\partial_t f_0(t,v) = \frac{e^2}{m^2}\sum_k |E_k(t)|^2 \partial_v \Big[\int_0^t dt' \exp\Big(ikv(t'-t) - i\int_t^{t'} dt''\, \omega_k(t'')\Big)\partial_v f_0(t',v) + c.c.\Big] \,, \qquad (5)$$

$$\partial_t |E_k|^2 = \frac{2\pi e^2 \omega_p}{mk} |E_k(t)|^2 \Big[\int_{-\infty}^{\infty} dv \int_0^t dt' \exp\Big(ikv(t'-t) - i\int_t^{t'} dt''\, \omega_k(t'')\Big)\partial_v f_0(t',v) + c.c.\Big] \,. \qquad (6)$$

A - External spectrum sampling (ES) Eq.(4) can be integrated for a given spectral evolution extracted from simulations. Using $G_\ell = \int_0^\tau d\tau' e^{i\ell u\tau'} \bar{\phi}_\ell \partial_u \bar{f}_B$ and $\bar{G}_\ell = e^{-i\ell u\tau} G_\ell$, we get

$$\partial_\tau \bar{f}_B(\tau,u) = \partial_u \Gamma_{es}(\tau,u) = \partial_u \sum_\ell \ell^2 (\bar{\phi}_\ell \bar{G}_\ell^* + \bar{\phi}_\ell^* \bar{G}_\ell) \,, \quad \partial_\tau \bar{G}_\ell(\tau,u) = -i\ell u \bar{G}_\ell + \bar{\phi}_\ell \partial_u \bar{f}_B \,. \qquad (7)$$

A $4th$ order Runge-Kutta algorithm evolves the system with $\bar{f}_B(0,u) = \bar{F}_B(u)$ and $\bar{G}_\ell(0,u) = 0$. Eq.(7) represents the first level of the hierarchy scheme: the fluxes Γ_{es} can be now evaluated by sampling only $\bar{\phi}_\ell(\tau)$ from Eq.(1) obtaining an approximated evolution of $\bar{f}_B(\tau,u)$.

B - Quasi-linear model (QL) The QL model, due to the specific underlying assumptions, corresponds to the second hierarchy level of the approximation scheme. The model results in a system of self-consistent equations for the DF evolution (no sampling from simulations). Eqs.(5)-(6) can be reduced using the following assumptions: quasi-stationarity of ω_k and f_0; marginal stability for $\mathrm{Im}(\omega_k) \ll \omega_p$; broad and dense spectrum, i.e. continuous k-space $k = \omega_p/v$ (we can use $E(t,k) \to E(t,v)$). Introducing the spectral function $\mathscr{I}(\tau,u) = |\bar{\phi}|^2$ (with $\mathscr{I}_0 = \mathscr{I}(0,u)$) and $\bar{\mathscr{N}} = M/(\ell_{max} - \ell_{min})$, $H(\tau,u) = (\pi\eta u^2/S)\int_0^\tau \partial_u \bar{f}_B d\tau'$, QL equations write

$$\partial_\tau \bar{f}_B(\tau,u) = \partial_u \Gamma_{ql}(\tau,u) = \partial_u \Big[\pi \bar{\mathscr{N}}\, \partial_u \bar{f}_B\, \mathscr{I}_0\, \exp[H]/u^3\Big] \,, \quad \partial_\tau H(\tau,u) = \pi\eta u^2\, \partial_u \bar{f}_B / S \,. \qquad (8)$$

Initial conditions are $\bar{F}_B(u)$, $H(0,u) = 0$ and the spectral evolution reads $\mathscr{I}_{QL}(\tau,u) = \mathscr{I}_0\, \exp[H]$.

C - Extension of QL model The QL system can be re-derived [4] using an expansion for the DF at short times. This formally extend the validity of the QL model to the temporal mesoscales before saturation getting the following spectral correction:

$$\mathscr{I} = \mathscr{I}_{QL}\big(|\partial_u \bar{F}_B|/|\partial_u \bar{f}_B|\big)^{\alpha}\,, \qquad \alpha = 4(\sqrt{2}-1)/\pi \simeq 0.51\,. \tag{9}$$

We recognize this model as a level 2.0 of the hierarchy scheme. Here, the spectral correction (Eq.(9)) is evaluated by using $\bar{f}_B$ sampled from simulations of the N-body scheme Eq.(1).

Numerical results We set a reference case of a Gaussian beam and 60 modes corresponding to a scenario with Kubo number $\mathscr{K} = \tau_{ac}/\tau_b \sim 0.02$. Simulations of Eq.(1) are in Fig.1 outlining the avalanche excitation of linear stable modes and the profile flattening (Fig.2, left).

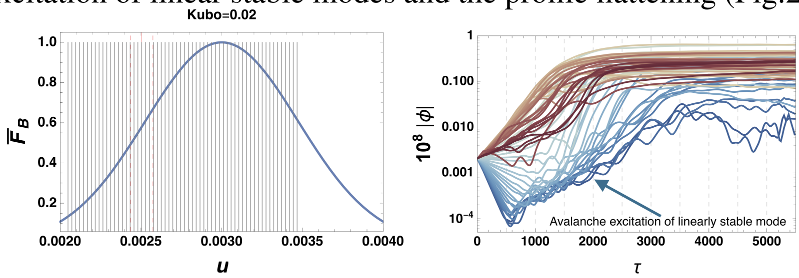

Figure 1: *Left: initial profile and resonances. Right: mode evolution from Eq.(1) (linear stable modes in blue).*

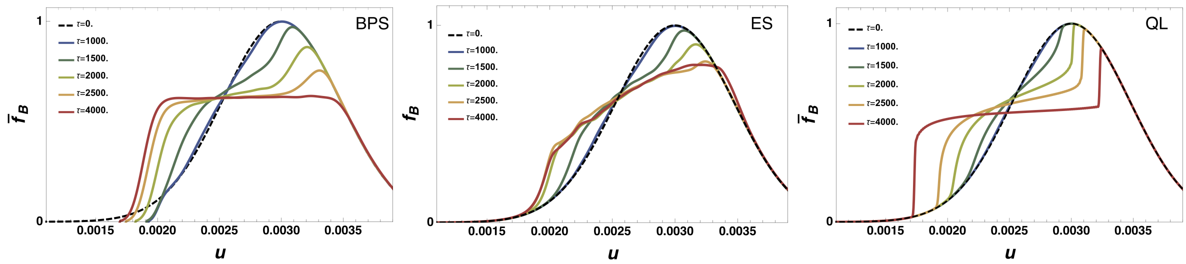

Figure 2: *DF evolution for: level 0, Eq.(1) (left); level 1, Eq.(7) (center); level 2, Eq.(8) (right).*

The evolution of $\bar{f}_B$ for the various approximation levels is depicted in Fig.2. The ES scheme (diagonal reduction) well reproduces the dynamics until saturation time scale, then mode-mode interaction ($k' \neq k$) becomes relevant and it loses predictivity. Regarding the QL evolution, we instead observe a retarded flattening formation, while the asymptotic plateau is well outlined.

The fluxes Γ_{bps} (level 0, Eq.(3)), Γ_{es} (level 1, Eq.(7)) and Γ_{ql} (level 2, Eq.(8)) are evaluated at different times (Fig.3). Since fluxes correspond to the DF drive, the properties of f_B discussed above are reflected in their evolution. In fact, for saturation time scales, the ES approximation well matches the self-consistent fluxes, while after saturation ($\tau \geq 2000$) it loses predictivity and the QL model starts to be comparable to the N-body simulations.

We conclude by plotting the spectral evolution compared to the mode evolution of the self consistent simulations (Fig.4). The QL model (Eq.(8)) is not predictive for the temporal mesoscales due to the non-pure diffusive character of the transport, while it properly envelope the discrete spectrum for late stages. Moreover, plotting the first order QL spectral extension (Eq.(9))

for the late linear phase, we outline how it properly enhances the instantaneous growth rate curing the mesoscale spectral evolution in the pre-saturation regime.

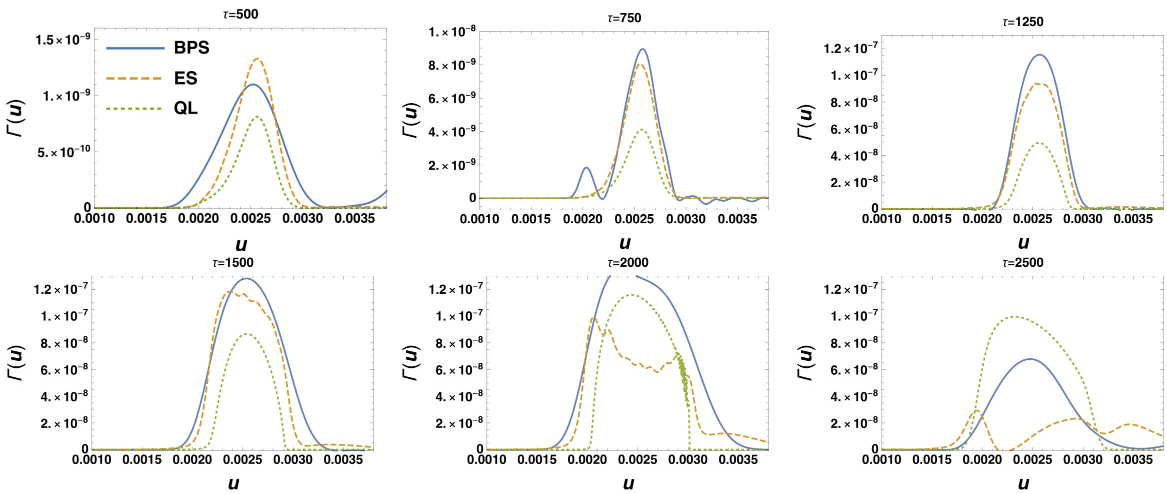

Figure 3: *Flux evolution (Γ_{bps}, Γ_{es} and Γ_{ql}) for the different levels of approximation.*

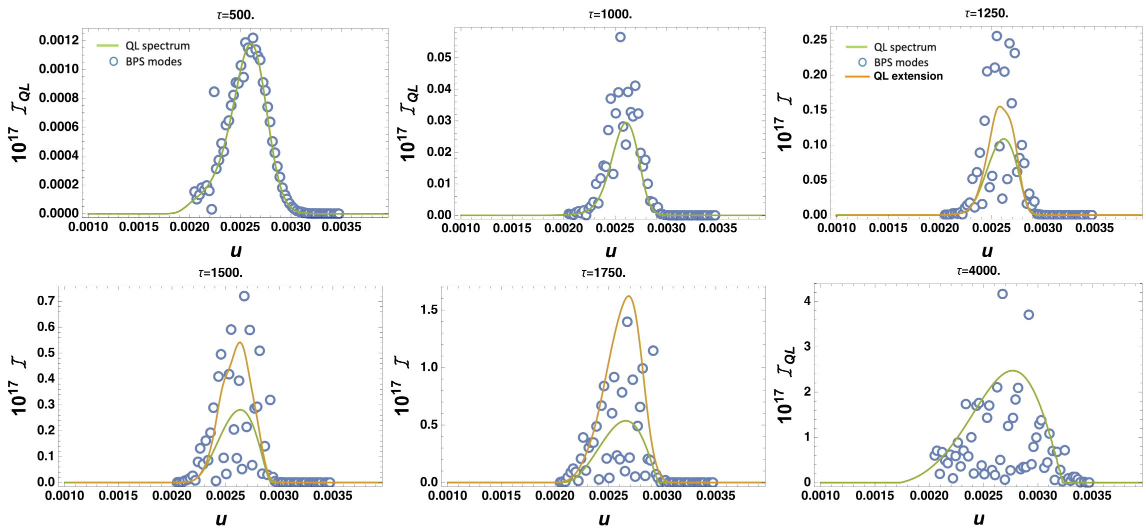

Figure 4: *QL spectral evolution from Eq.(8) (green) and from Eq.(9) (orange). Bullets are modes from Eq.(1).*

Concluding remarks Our analysis, based on N-body simulations as reference term for establishing the predictivity of different VP equation approximations, fixed a precise hierarchy related to different time scales. While the QL model is predictive in the late evolution, it fails in the temporal meso-scale, where the diagonal VP formulation appears as very reliable to account for the spectrum saturation. The latter is, thus, the most appropriate paradigm when the isomorphism between the BPS and the fast ions interacting with with Alvénic modes is implemented.

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