Stability of electrostatic axisymmetric perturbations in rotating Hall plasmas

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Plasma immersed in crossed electric $\mathbf{E}_0$ and magnetic $\mathbf{B}$ fields is subjected to a lot of instabilities associated with $\mathbf{E}_0 \times \mathbf{B}$ rotation [1, 2]. The most common of them are of a convective nature and occur for non-axisymmetric flute-like perturbations with $m \neq 0$ and $k_{\parallel} = 0$ (m is the azimuthal wave-number of perturbations and $k_{\parallel}$ is the projection of the wave vector on the direction of magnetic field). This paper analytically investigates the stability of a differentially rotating plasma in crossed electric and magnetic fields. The equation, describing the electrostatic axisymmetric ($m = 0$) perturbations of the above plasma state, is derived within the framework of a two-fluid model. The necessary and sufficient instability conditions as well as the growth rate of unstable modes are obtained. It is shown that instability arises if the generalized momentum of ions decreases with radius.

We study the stability of axisymmetric electrostatic perturbations in a quasineutral plasma column immersed in crossed magnetic $\mathbf{B}$ and electric $\mathbf{E}_0$ fields, which is shown in Fig. 1. The magnetic field is axial and homogeneous $\mathbf{B} = B\mathbf{e}_z$, $B = \text{const}$. The electric field $\mathbf{E}_0 = E_0(r)\mathbf{e}_r$ is radial and depends on radius. The radial profile of electric field can be maintained by a set of annular biased electrodes at different potentials $\phi = \phi_1 ... \phi_N$. The equilibrium plasma density also depends only on the radial coordinate, $n_0 = n_0(r)$. We consider rotating plasma with hot electrons and cold ions in crossed electric and magnetic fields described by the following set of two-fluid equations:

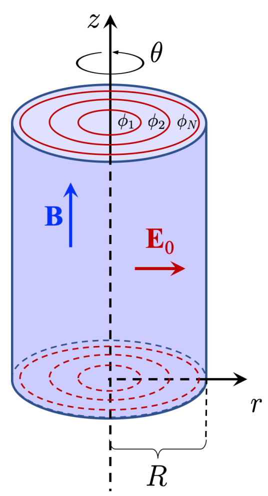

Figure 1: Geometry of the considered system.

$$\frac{\partial \mathbf{v}_i}{\partial t} + (\mathbf{v}_i \cdot \nabla)\mathbf{v}_i = \frac{Ze}{m_i}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_i \times \mathbf{B}\right), \tag{1}$$

$$\frac{\partial n_i}{\partial t} + \operatorname{div}(n_i \mathbf{v}_i) = 0, \tag{2}$$

$$e\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_e\times\mathbf{B}\right)+\frac{T_e}{n_e}\nabla n_e=0. \tag{3}$$

Here $\mathbf{v}_j,\ n_j$ are the velocity and density of plasma species, $j=(i,e)$, $Zn_i=n_e$; T_e is the electron temperature, which is assumed to be constant, $T_e=\text{const}$; m_i,Z,e,c are the ion mass, the charge number, the elementary charge and the speed of light, respectively. Hereafter and below the CGS units are used.

In a stationary state, the velocity of ions $\mathbf{v}_{i0}=v_0(r)\mathbf{e}_\theta$ satisfies the condition of the radial force balance given by Eq. (1) and is described by the expression

$$v_{0\pm}=-\frac{r\omega_{Bi}}{2}\left[1\pm\sqrt{1+4\frac{V_E}{r\omega_{Bi}}}\right]. \tag{4}$$

Here $V_E=-cE_0/B$ is the velocity of $\mathbf{E}_0\times\mathbf{B}$-drift, $\omega_{Bi}=ZeB/m_ic$ is the ion cyclotron frequency. Below we consider only the slow rotation mode $v_0\equiv v_{0-}$, for which the ion velocity is equal to zero in the absence of electric field. Equation (4) shows that for a positive electric field $E_0>0$ (directed from $r=0$ to $r=R$) the equilibrium plasma state exists only for

$$E_0<E_{cr},\quad E_{cr}=\frac{r\omega_{Bi}^2}{4}\frac{m_i}{Ze}. \tag{5}$$

In the limit $E_0\to E_{cr}$ the velocity of ions tends to the critical value $v_0\to v_{0cr}\equiv -r\omega_{Bi}/2$, which is usually called the *Brillouin limit* [3].

The equilibrium rotation of electrons $\mathbf{v}_{e0}=u_0(r)\mathbf{e}_\theta$ given by Eq. (3) consists of electric V_E and diamagnetic $V_{\star e}$ drifts:

$$u_0=V_E+V_{\star e}, \tag{6}$$

where $V_{\star e}=(cT_e/eB)d\ln n_0/dr$.

For electrostatic axisymmetric perturbations, $\mathbf{E}=-\nabla\tilde{\phi}(\mathbf{r},\mathbf{z},\mathbf{t})$, with the perturbed plasma potential in the form $\tilde{\phi}(r,z,t)=\Phi(r)\exp[i(k_\parallel z-\omega t)]$ the system of equation (1)-(3) results in the following differential equation for function Φ:

$$\frac{1}{rn_0}\left(\frac{c_s^2}{\omega^2-\kappa\omega_{Bi}^2}rn_0\Phi'\right)'+\left(1-\frac{k_\parallel^2c_s^2}{\omega^2}\right)\Phi=0, \tag{7}$$

where $c_s^2=ZT_e/m_i$ is the ion-sound speed, prime implies the radial derivative $d(...)/dr$ and

$$\kappa=\left(1+2\frac{\Omega}{\omega_{Bi}}\right)\left(1+\frac{1}{r}\frac{(r^2\Omega)'}{\omega_{Bi}}\right). \tag{8}$$

Together with the boundary conditions, Eq. (7) constitutes the eigenvalue problem. For plasma column with ideally conducting wall at radius R, we require $\Phi(r=R)=0$, which provides zero tangential component of the perturbed electric field, $\tilde{E}_z=-k_\parallel\Phi$. At the center of plasma column, $r=0$, the solution is required to be finite, $|\Phi(0)|<\infty$.

In the local approximation, the radial dependence of the perturbed plasma potential in Eq. (7) can be written in the Fourier form $\Phi(r) \sim \exp(ik_r r)$ with the radial wave-number k_r, which results in the local dispersion relation

$$1 - \frac{k_\parallel^2 c_s^2}{\omega^2} - \frac{k_r^2 c_s^2}{\omega^2 - \kappa\omega_{Bi}^2} = 0. \tag{9}$$

Note that the local dispersion relation does not include the density profile $n_0(r)$.

The solutions of Eq. (9) have the form

$$\omega_\pm^2 \;=\; \frac{1}{2}(\omega_s^2 + \kappa\omega_{Bi}^2) \pm \frac{1}{2}\sqrt{(\omega_s^2 + \kappa\omega_{Bi}^2)^2 - 4\kappa\frac{k_\parallel^2}{k^2}\omega_s^2\omega_{Bi}^2}, \tag{10}$$

where $\omega_s = kc_s$ is the ion-sound frequency and $k = \sqrt{k_\parallel^2 + k_r^2}$. With no rotation $\kappa = 1$ ($\Omega = 0$) Eq. (10) describes the dispersion of ion-sound modes in magnetized plasma. Waves with frequencies ω_+ and ω_- are usually called the *fast* and *slow* ion-sound modes, respectively. The finite speed of plasma rotation destabilizes the slow ion-sound mode if $\kappa < 0$. In physical variables this instability condition has the form

$$1 + \frac{(r^4\Omega_E)'}{r^3\omega_{Bi}} < 0, \tag{11}$$

where $\Omega_e = V_E/r$ is the angular velocity related to $\mathbf{E}_0 \times \mathbf{B}$-drift, or using the definition of the generalized momentum $P_\theta = m_i r v_0 + (Ze/c) r A_\theta$ ($A_\theta = Br/2$ is the vector potential for constant magnetic field),

$$\frac{\partial P_\theta}{\partial r} < 0.$$

The growth rate $\gamma = \mathrm{Im}(\omega)$ of the unstable slow ion-sound mode ω_- is given by

$$\gamma^2 = \frac{\omega_{Bi}^2}{2}\left[\sqrt{(k^2\rho_s^2 - |\kappa|)^2 + 4\lambda^2|\kappa|k^2\rho_s^2} - (k^2\rho_s^2 - |\kappa|)\right], \tag{12}$$

where $\lambda = k_\parallel/k$ and $\rho_s = c_s/\omega_{Bi}$ is the ion-sound Larmor radius. The analysis of Eq. (12) shows that the growth rate has a maximum $\gamma = \omega_{Bi}\sqrt{|\kappa|}$ at $k_r \to 0$; at finite k_r the growth rate increases with $k_\parallel$ – see fig. 2 (solid lines).

In the limiting case $k^2\rho_s^2 \gg |\kappa|$ Eq. (12) reduces to

$$\gamma^2 \approx \lambda^2\omega_{Bi}^2|\kappa|, \tag{13}$$

which conforms with incompressible perturbations. The dependence of γ on $k_r\rho_s$ described by Eq. (13) is shown in Fig. 2 by dashed lines.

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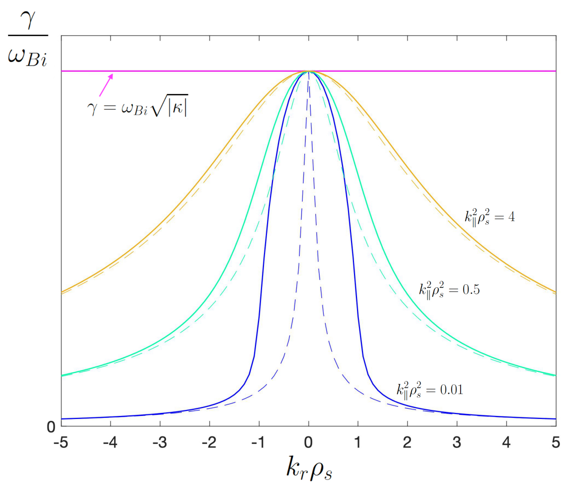

Figure 2: Dependencies of the growth rate γ on radial wave-number $k_r\rho_s$ at different values of axial wave-number $k_\parallel\rho$; $|\kappa| = 1$. The straight lines and the dashed lines correspond to Eq. (12) and Eq. (13), respectively.

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