Theory of Vertical Displacements resonant at magnetic divertor X-points

F. Porcelli[1], A. Yolbarsop[2], T. Barberis[1] and R. Fitzpatrick[3]

[1]*Dept. of Applied Science and Technology, Polytechnic University of Turin, Italy*

[2]*KTX Laboratory, School of Nuclear Science and Technology, USTC, Hefei, 230022, China*

[3]*Institute for Fusion Studies, University of Texas at Austin, USA*

Vertical displacements are axisymmetric modes in tokamak plasmas. They are resonant in two different ways. First, toroidal magnetic field lines going through the X-points of a divertor separatrix are resonant to n=0 MHD perturbations [1], as the resonant condition $\mathbf{B_{eq}} \cdot \nabla\xi = i(n/R)B_T\xi + iB_P\sum(m/r)\xi_m = 0$, with ξ a generic perturbation, is satisfied for toroidal mode number $n = 0$ and any poloidal mode number m, since the equilibrium poloidal magnetic field B_P vanishes at the X-point. In this case, however, the MHD resonance condition is satisfied on a magnetic field line, and not on a mode-rational magnetic surface, which requires a non-standard treatment of the ideal-MHD resonance. Secondly, two of the roots of the relevant dispersion relation correspond to oscillatory modes, weakly damped by wall resistivity, oscillating just below the poloidal Alfvén frequency. These modes are immune to continuum damping and can be driven unstable by mode-particle resonance with energetic fast ions, as we discussed in Ref. [3]. In this article, we are concerned only with the first type of resonance.

As is well known, vertical modes are ideal-MHD unstable in an elongated tokamak plasma, unless a nearby wall is present [2]. When the wall is ideal and the distance between the plasma boundary and the wall is relatively small, the ideal-MHD vertical instability is suppressed by image currents that are induced on the wall. If the wall is resistive, the relevant dispersion relation is cubic. Apart from the two damped oscillatory solutions that have been mentioned above, the third root is purely growing on the resistive wall time scale. This slow growth can be suppressed by an active feedback stabilization system. The standard procedure adopted by active feedback controllers is to treat the displaced plasma as a sequence of ideal-MHD equilibria. Then, electromagnetic forces between the displaced plasma and active feedback external currents are calculated in order to control the plasma vertical position. However, in doing so, plasma current sheets that can be induced near the magnetic X-points are not taken into account. This is the main effect that we address in this article. The analysis is based on the reduced ideal-MHD model (RIMHD). For reasons of brevity, we skip the mathematical details, which can be found in [1], and summarize the main points. We remark that, in the following, the no-wall case is considered. Nevertheless, and perhaps surprisingly, we will find that vertical displacements are ideally stable even in the absence of a wall, if the X-point resonance is taken into account.

For the sake of analytic work, a simple equilibrium is chosen, such that the equilibrium plasma current is uniform up to a convenient elliptical flux surface, where $\mu = \mu_b$, and zero outside, while the plasma density is uniform and extends all the way to the magnetic separatrix. When elliptical coordinates (μ, θ) are used, the linearized RIMHD equations yield solutions that correspond to rigid vertical shifts of the plasma column, involving perturbations with a single $m = 1$ Fourier component in the elliptical angle θ. The perturbed stream function corresponding to the rigid shift solution can be written as

$$\tilde{\varphi}(\mu, \theta) = \gamma \xi A \sinh\mu \cos\theta, \tag{1}$$

where ξ is a constant parameter representing the vertical shift of the magnetic axis, $A = \sqrt{b^2 - a^2}$ and γ is the growth rate. The solution for the perturbed flux for $\mu < \mu_b$ is

$$\tilde{\psi}(\mu, \theta) = -\frac{\xi}{b} \frac{\cosh\mu}{\cosh\mu_b} \sin\theta, \tag{2}$$

and for $\mu > \mu_b$ all the way up the magnetic separatrix (X-points are at $\mu = 2\mu_b$ and $\theta = \pi/2 \pm n\pi$),

$$\tilde{\psi}(\mu, \theta) = \frac{\xi}{b} \frac{\sinh(\mu - 2\mu_b)}{\sinh\mu_b} \sin\theta. \tag{3}$$

On the other hand, the magnetic separatrix, where the resonant X-points lie, is not a μ =const elliptical surface, and therefore the solution beyond the separatrix couples several θ harmonics. It is then convenient to introduce flux coordinates for values of $\mu > \mu_b$,

$$u = \alpha^{-2}[\psi_{eq}(\mu, \theta) - \psi_X] \tag{4}$$

$$v = \theta - \frac{\pi}{2} + \frac{e_0}{2} \cosh[2(\mu - \mu_b)] \sin(2\theta), \tag{5}$$

where ψ_{eq} is the equilibrium flux function and ψ_X its value on the magnetic separatrix, $e_0 = (b^2 - a^2)/(b^2 + a^2)$ is the ellipticity parameter (with a and b the minor and major semiaxes of the $\mu = \mu_b$ surface), and $\alpha^2 = (1 - e_0^2)^{-1/2}$. The magnetic separatrix corresponds to $u = 0$ and the X-points have coordinates $u = 0$, $v = 0 \pm n\pi$.

A complete analytic solution of the linearized RIMHD model can be obtained using flux coordinates. In particular, the perturbed flux on the magnetic separatrix behaves as

$$\tilde{\psi}(0, v) = \frac{\xi}{b} \sum_{m,odd}^{\infty} \alpha_m \cos(mv), \tag{6}$$

where the Fourier coefficients α_m are fully determined [1]. It is found that these coefficients do not decay exponentially with m at large values of m, which is indicative of singular behavior. The asymptotic behaviour at large m is $\alpha_m \sim p/m^{3/2}$, where $p = \alpha[(a^2 + b^2)/\pi a^2]^{1/2} e_0^{1/2}$ is a positive constant. The corresponding behavior of $\tilde{\psi}(0,v)$ approaching the X-point at $v=0$ is $\tilde{\psi}_\Delta(0,v) \sim -p(\pi v/2)^{1/2}$, revealing a square-root singularity. Another technical issue that was addressed and solved in [1] is that the coordinate v is no longer periodic beyond the separatrix, where the behaviour of the perturbed flux is anyway needed in order to evaluate the current sheets that form on the separatrix. After not-so-straightforward algebra, whose details can be found in [1], we obtain the following expression for the perturbed current density sheet on the magnetic separatrix, $\tilde{J}(u,v) = j_X(v)\delta(u)$, in the vicinity of the magnetic X-point at $v=0$:

$$j_X(v) \sim -\frac{2e_0}{ab}\sqrt{\frac{\pi}{2}}\left(\frac{p}{2}+q\right)\frac{\xi}{b}|v|^{1/2} \quad \text{as} \quad v \to 0, \tag{7}$$

where the parameter $q = p/2$ for modes that are even in v, and $q = -p/2$ for modes that are odd in v. Thus, for the latter modes, the current sheet actually vanishes. The corresponding dispersion relation is quadratic in $\gamma = -i\omega$:

$$\gamma^2 = -2\sqrt{\frac{\pi a}{2b}}\left(q+\frac{p}{2}\right)\left(1-e_0^2\right)^{1/2} e_0^{3/2}\omega_A^2 \tag{8}$$

where $\omega_A = B_p'/(4\pi\rho_m)^{1/2}$ is the relevant Alfvén frequency, with B_p' the on-axis radial derivative of the equilibrium poloidal field. Thus, for even modes, γ^2 is negative, which implies stable oscillatory solutions even in the absence of a nearby wall, as we anticipated. For odd modes, $\gamma^2 = 0$ and vertical displacements are neutrally stable. These statements are of course true in the ideal-MHD limit.

Thus, the relevant result is that X-points are capable of suppressing the vertical instability developing on ideal-MHD time scales, even in the absence of a nearby wall. The stabilization mechanism is a direct consequence of the ideal-MHD flux-freezing constraint on the X-points, which generates current sheets localized along the magnetic separatrix, exerting a force capable of pushing back the plasma in its vertical motion. Analogies with the physics of current sheet formation for the nonlinear evolution of internal kink modes [4] and with the island coalescence problem [5] come to mind. In these cases, the ideal-MHD constraint causes magnetic flux to pile up near the X-points, leading to perturbed localized currents and a stabilizing effect in the ideal-MHD limit. One question is whether these X-point current sheets have been observed experimentally. Although the X-point is a difficult region to diagnose properly, Refs. [6] and [7] suggest that indeed axisymmetric X-point currents have been detected experimentally, although in circumstances not directly related to the vertical instability. Also, axisymmetric X-point current sheets have been found in numerical simulations [8].

In conclusion, we have discussed here the first step in the development of the analytic theory of resonant axisymmetric modes in tokamak plasmas. Since the n=0 resonance at divertor

X-points is an ideal-MHD phenomenon, it is reasonable that it must be treated first according to the ideal-MHD model. Hybrid kinetic-MHD effects related to the resonant interaction of oscilatory n=0 modes with fast ions were studied in Ref. [3]. Future work will consider resistive and extended-MHD effects. Here, we have found that, when the plasma density extends to the magnetic separatrix and $n = 0$ perturbations resonate at the magnetic X-points, vertical displacements are stable, at least on ideal-MHD time scales, without any need for passive stabilization elements. The stabilization mechanism is a direct consequence of the ideal-MHD flux-freezing constraint on the X-points, which generates current sheets localized along the magnetic separatrix, exerting a force capable of pushing back the plasma in its vertical motion. This also suggests that plasma electrical resistivity in a narrow boundary layer along the magnetic separatrix, in addition to wall resistivity, may have a profound impact on the stability of $n = 0$ vertical displacements, and we suspect that specific n=0 resistive modes localized near the X-point may be excited [9].

We observe that the X-point resonance is relevant not only to vertical displacements, but to any circumstance where $n = 0$ perturbations may develop, whether linearly or nonlinearly, in a tokamak plasma. One example is the use of "vertical kicks" for the mitigation of ELMs. Also, the nonlinear evolution of peeling-ballooning modes is likely to produce a n=0 component that resonate at the X-point, giving rise to axisymmetric current sheets as observed on JET during type-I giant ELMs and described in [6]. In turn, axisymmetric current sheets driven at magnetic X-points and along the last closed flux surface may affect the stability of peeling modes. We may speculate that the formation of current sheets extended along the divertor separatrix in the vicinity of the X-points tends to be an ubiquitous phenomenon in tokamak plasma with divertors, although we are not able to provide at this stage, from a simple analytic point of view, an estimate of the amplitude of these current sheets, as this will require nonlinear work.

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