Systematic dissection of the guiding center phase space based on orbital spectrum analysis

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Introduction

The Guiding Center (GC) description of plasma particle motion for an axisymmetric toroidal magnetic field configuration allows for the analytical calculation of the full Orbital Spectrum for a large aspect ratio equilibrium magnetic field, in terms of the three Constants Of the Motion (COM). In this work, the analytical results are utilized to construct the full skeleton of the resonance structure in the space of the COM, along which significant particle energy and momentum transport takes place under the presence of non-axisymmetric perturbations. Moreover, we pinpoint the location and the extent of various resonances in the GC phase space [2] and systematically dissect the phase space with appropriate Poincare surfaces of section to confirm our analytical predictions.

Guiding Center Hamiltonian

A general axisymmetric toroidal magnetic configuration consisting of nested toroidal flux surfaces can be represented in White–Boozer [1] coordinates as $B = g(\psi)\nabla\zeta + I(\psi)\nabla\theta + \delta(\psi,\theta)\nabla\psi_p$ where ζ and θ are the toroidal and poloidal angles. The toroidal flux ψ is related to the poloidal flux ψ_p through the safety factor $q(\psi) = d\psi/d\psi_p$. The functions g and I are related to the poloidal and toroidal currents and δ is related to the non-orthogonality of the coordinate system. The GC motion of a charged particle is described by the Hamiltonian $H = \rho_{||}^2\boldsymbol{B}^2/2 + \mu\boldsymbol{B}$, where $\boldsymbol{B}$ is the magnetic field, μ is the magnetic moment and $\rho_{||}$ is the velocity component parallel to the magnetic field. The three couples of canonical conjugate variables for this GC Hamiltonian are (μ,ξ), (P_θ,θ) and (P_ζ,ζ) with $P_\theta = \psi + \rho_{||}I(\psi)$ and $P_\zeta = \rho_{||}g(\psi) - \psi_p$ [1]. The Hamiltonian can be written with respect to these canonical variables, as $H(P_\theta,\theta,P_\zeta,\zeta,\mu,\xi)$.

A general canonical transformation to drift orbit deviation variables transforms the above Hamiltonian to a new (barred) variable set [2]. The physical meaning of the new canonical variables becomes obvious for a Large Aspect Ratio (LAR) cylindrical equilibrium described by $g = 1$, $I = 0$, and $B = 1 - r\cos\theta$, where $r = \sqrt{2\psi}$ [1]. In this case the initial variables take the form $P_\theta = \psi$, $P_\zeta = \rho_{||} - \psi_p(\psi)$ and the barred variables $\bar{P}_\theta = \psi - \psi_0$, $\bar{\theta} = \theta - q^{-1}(\psi)$,

$\bar{P_\zeta} = P_\zeta + \psi_p(\psi), \bar{\zeta} = \zeta$ [2]. Evaluation of the magnetic field on a particular magnetic surface of reference, $B(\psi, \theta) \to B(\psi_0, \theta)$, whereas the GC deviation from a field line and particle drifts are possible, allows for the analytical calculation of the full Orbital Spectrum. The magnetic surface of reference ψ_0 can be written in terms of the Constants Of the Motion (COM) variables $(E,\ \mu,\ P_\zeta)$ [3]. The corresponding configuration space is depicted in Fig. 1.

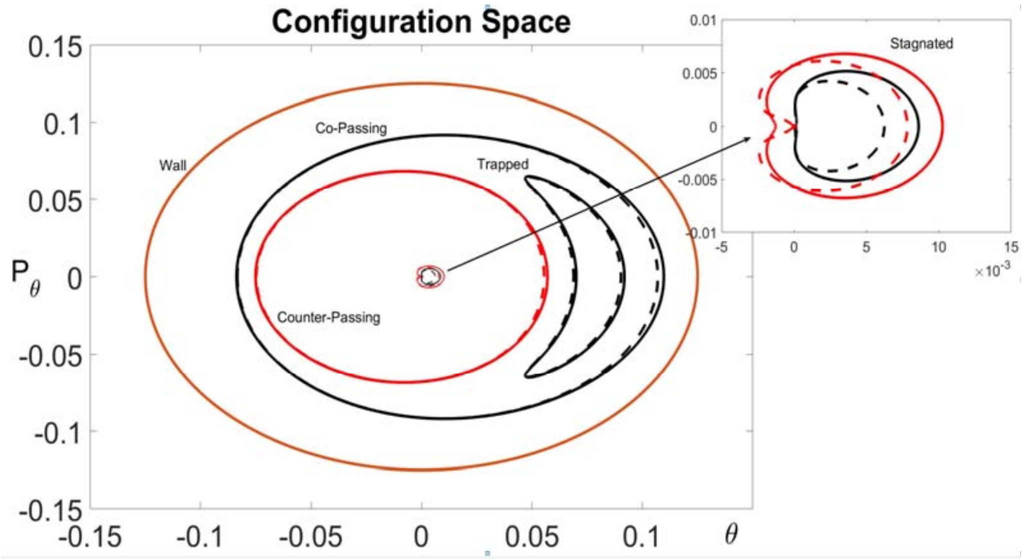

Figure 1: The projection on (P_θ, θ) plane of the trajectories in configuration space for fixed value of $\mu = 1.27 \cdot 10^{-4}$. Orange line depicts the maximum magnetic surface, which for the case of a LAR equilibrium equals $r = 0.5$. Solid lines are the trajectories for the Full motion whereas dash lines are the trajectories when the magnetic field is evaluated on a magnetic surface of reference ψ_0 . Co-Passing, Counter-Passing and Trapped orbits were obtained for $P_\zeta = -0.08$. Stagnated (potato) orbits were obtained for $P_\zeta = -0.002$. The magnetic surface of reference have been chosen to be $\psi_0 = -P_\zeta$ for Trapped, $\psi_0 = -P_\zeta - \sqrt{2(E-\mu)}$ for Counter passing and $\psi_0 = -P_\zeta + \sqrt{2(E-\mu)}$ for Co-Passing orbits respectively [3]. Significant agreement is observed between the exact trajectories (solid lines) and the trajectories with the magnetic field evaluated at ψ_0.

Analytical calculation of the Actions and the Orbital Frequencies

The action-angle transformation $\bar{H}(\bar{P_\zeta}, \bar{\zeta}, \bar{P}_\theta, \bar{\theta}, \bar{\mu}, \bar{\xi}) \leftrightarrow \hat{H}(J_\zeta, J_\theta, J_\xi)$ allows for the analytical calculation of the orbital frequencies for the three degrees of freedom. Therefore:

$$\widehat{\omega}_\zeta = \frac{\partial \hat{H}}{\partial J_\zeta}, \qquad \widehat{\omega}_\theta = -\widehat{\omega}_\zeta \frac{\partial J_\zeta}{\partial J_\theta}, \qquad \widehat{\omega}_\xi = -\widehat{\omega}_\zeta \frac{\partial J_\zeta}{\partial J_\xi}$$

where $\widehat{\omega}_\zeta$ is the bounce/transit frequency, $\widehat{\omega}_\theta$ is the bounce/transit-averaged toroidal precession frequency, $\widehat{\omega}_\xi$ is the bounce/transit-averaged gyration frequency and $(J_\zeta, J_\theta, J_\xi)$ are the actions. The three actions and the bounce (b) / transit (t) frequencies are given by the analytical expressions

$$J_\zeta^b = \frac{8q(\psi_0)\sqrt{\mu r}}{\pi\eta(1-r)}[(\eta k - 1)\Pi(\eta k, k) + K(k)],\ J_\zeta^t = \frac{4q(\psi_0)\sqrt{\mu r}}{\pi\eta(1-r)\sqrt{k}}[(\eta k - 1)\Pi(\eta, k^{-1}) + K(k^{-1})]$$

$$J_\theta^{b,t} = -q(\psi_0)\left(P_\zeta + \psi_p(\psi_0)\right), \quad J_\xi^{b,t} = \mu, \quad \widehat{\omega}_\zeta^b = \frac{\pi(1-r)\sqrt{\mu r}}{2q(\psi_0)\Pi(\eta k,k)}, \quad \widehat{\omega}_\zeta^t = \frac{\pi\sqrt{k}(1-r)\sqrt{\mu r}}{2q(\psi_0)\Pi(\eta,k^{-1})}$$

where $r = \sqrt{2\psi_0}$, $k = \frac{E-\mu(1-r)}{2\mu r}$, $n = -\frac{2r}{1-r}$. The analytically expressions for the frequencies $\widehat{\omega}_\theta$ and $\widehat{\omega}_\xi$ are too lengthy to be given here. The expression of the magnetic surface of reference ψ_0 in terms of the COM variables $(E,\ \mu,\ P_\zeta)$ enables us to express the Actions in terms of the constants of the motion of the initial system, thereby the ratio $\frac{\widehat{\omega}_\theta}{\widehat{\omega}_\zeta} = -\frac{\partial J_\zeta}{\partial J_\theta}$ can be written as a function of $(E,\ \mu,\ P_\zeta)$. Resonance conditions correspond to rational values of this ratio in the COM space (Fig. 2).

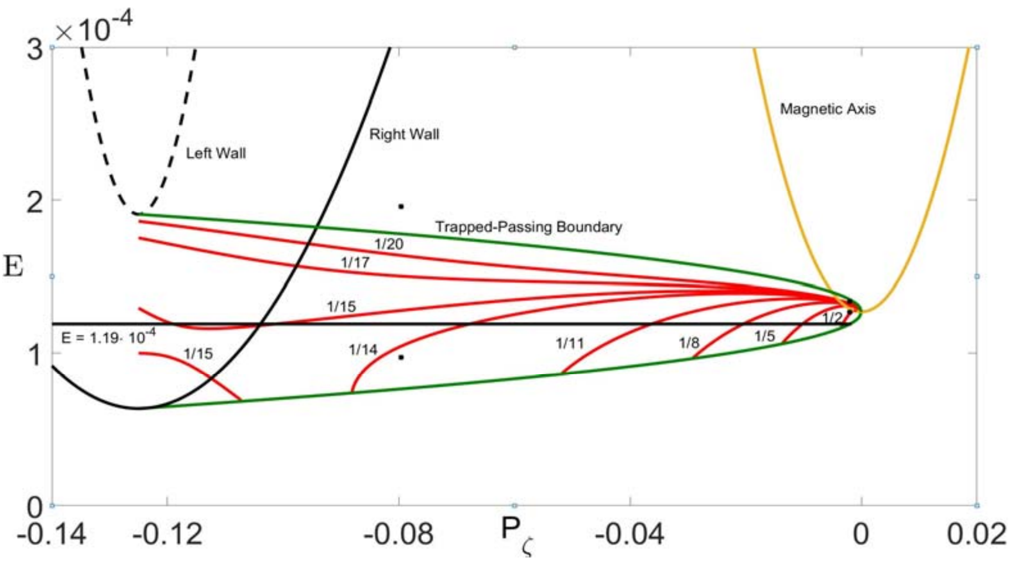

Figure 2: Red solid lines are the lines on (E, P_ζ) plane where the equation $\frac{\widehat{\omega}_\theta}{\widehat{\omega}_\zeta} = -\frac{\partial J_\zeta}{\partial J_\theta} = \frac{m}{n}$ is fulfilled, setting $\mu = 1.27 \cdot 10^{-4}$. Black dash and solid parabola lines depict the Left and Right Wall respectively, while orange solid line depicts the magnetic axis. The black solid straight line corresponds to a constant energy value, on this energy different resonances take place on different P_ζ values. Black dots on P_ζ = -0.08 denote the points on (E, P_ζ) plane where Trapped, Co-Passing and Counter-Passing orbits in Fig. 1 take place whereas black dots on $P_\zeta = -0.002$ denote the stagnated orbits for which we observe that they are very close to the magnetic axis.

Non-axisymmetric Perturbations and Resonance Conditions

The presence of non-axisymmetric perturbations results in a Hamiltonian of the form:

$$H = \widehat{H}\left(J_\zeta, J_\theta, J_\xi\right) + \sum_{m,n,l} H_{m,n,l}\left(J_\zeta, J_\theta, J_\xi\right) \exp\left[i(m\widehat{\theta} - n\widehat{\zeta})\right]$$

The perturbations affect particle and momentum transport in a resonant fashion. The interactions with perturbations take place in a constant energy surface. The resonance condition $m\widehat{\omega}_\theta - n\widehat{\omega}_\zeta = 0$ and the energy conservation condition $\widehat{H}\left(J_\zeta, J_\theta, J_\xi\right) = C$, writing the Actions in terms of $(E,\ \mu,\ P_\zeta)$, allow us to pinpoint the exact locations of resonances in the COM space as shown in Fig. 2. An illustration is given in Fig.3 where the Poincare plots constructed for the perturbed Hamiltonian:

$$H = H_0(\zeta, P_\zeta, \theta, P_\theta, \xi, \mu) + \varepsilon \cdot \sin(m \cdot \theta - n \cdot \zeta)$$

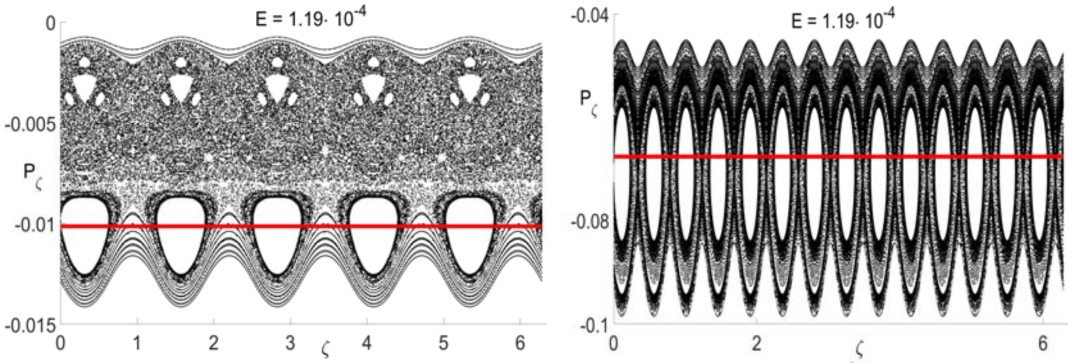

Figure 3: Left, the Poincare plot for m=1, n=5. Right, the Poincare plot for m=1, n=14. Both plots correspond to the constant energy $E = 1.19 \cdot 10^{-4}$ and $\mu = 1.27 \cdot 10^{-4}$. We observe that on P_ζ values which are indicated in Fig. 2 by the intersection points of the black solid straight line $E = 1.19 \cdot 10^{-4}$ with the red solid lines of resonances 1/5 and 1/14 the resonant islands in Poincare plots indeed appear.

Summary and Conclusions

Evaluation of the magnetic field on a particular magnetic surface of reference leads to analytical expressions for the frequencies and show a remarkable agreement with numerically calculated frequencies. The Action-Angle transformation allows for determining the resonance conditions under particle interaction with non-axisymmetric perturbations that affect energy, momentum and particle transport in toroidal plasma configurations and the application of standard canonical perturbation methods as well as the systematic dynamical reduction and the formulation of a bounce-kinetic description.

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[2] Y. Antonenas, G. Anastassiou, and Y. Kominis, *Analytical Calculation of the Orbital Spectrum of the Guiding Center Motion in Axisymmetric Magnetic Fields,* J. Plasma Phys. 87, 855870101 (2021).

[3] S.D. Pinches et al, *The HAGIS self-consistent nonlinear wave-particle interaction model*, Computer Physics Communications 111, 133-149 (1998)

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