On Relativistic Braginskii Transport Equations: Mixed Approach

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In recent decades it has been shown [1, 2, 3, 4] that the relativistic effects in fusion plasmas with temperatures of tens of keV, despite $T_e \ll m_e c^2$, can produce non-negligible effects in transport. At the same time, nearly all predictions for fusion reactor scenarios, even for aneutron projects with $T_e \sim 100$ keV, have been made using nonrelativistic transport theory [5, 6].

In the present work, we consider a hot plasmas where the electrons with $u_{te} \sim c$ require a fully relativistic description (here, $u_{te} \equiv p_{te}/m_e = \sqrt{2T_e/m_e}$). We also assume that the hydrodynamic flows in such plasmas are slow compared to the thermal velocity, $V \ll u_{te}$ (runaway electrons are excluded from consideration) and such flows are considered in the weakly relativistic approach. Such consideration, fully relativistic for thermal plasma and weakly relativistic for fluxes, we call "mixed approach". In addition, improvements in the methods for solving the relativistic kinetic equation are proposed. Since the main goal of this paper is to include relativistic effects and to preserve the classical-like form of the equations, we do not use the 4-vector formalism.

Derivation of the relativistic Braginskii equations

To begin with, it is convenient to write a relativistic kinetic equation for the electron distribution function f_e in divergent form and without the 4-vectors,

$$\frac{\partial f_e}{\partial t} + \frac{\partial}{\partial x_k}(v_k f_e) + \frac{\partial}{\partial u_k}\left(\frac{e}{m}\left(E_k + \frac{1}{c}\left[\mathbf{v}\times\mathbf{B}\right]_k\right) f_e\right) = C_{ee}(f_e) + C_{ei}(f_e), \tag{1}$$

where v_k is velocity and $u_k = \gamma v_k$ is momentum per unit mass with $\gamma = \sqrt{1+u^2/c^2}$ as the relativistic factor. All other notations are standard.

As usual [1, 7], we assume that electrons are near thermodynamic equilibrium, $f_e = f_{e0} + f_{e1}$, where f_{e0} is "drifting" Maxwell-Jüttner distribution function, which can be represented as

$$f_{e0} = C_{MJ}\,\frac{n}{\pi^{3/2}u_{te}^3}\,\exp\left[-\mu\gamma_0\left(\gamma - \frac{1}{\gamma_0} - \frac{V_k u_k}{c^2}\right)\right], \tag{2}$$

with $\mu = mc^2/T \gg 1$, and $\gamma_0 = 1/\sqrt{1-V^2/c^2}$, which is the relativistic flow factor, related to the hydrodynamic flow. Since $V^2/c^2 \ll 1$, we take it below in the weakly relativistic limit, $\gamma_0 \simeq 1 + V^2/2c^2$. The normalizing coefficient equals

$$C_{MJ} = \sqrt{\frac{\pi}{2\mu}}\,\frac{\mathrm{e}^{-\mu}}{K_2(\mu)} = 1 - \frac{15}{8\mu} + O\left(\frac{1}{\mu^2}\right), \tag{3}$$

with $K_n(\mu)$ as the modified Bessel function of second kind of the n-th order.

To calculate moments in the local frame, let us first introduce the rest frame in which there are no hydrodynamic flows. In this case, Maxwell-Jüttner distribution function Eq. (2) should be taken with $V = 0$ and $\gamma_0 = 1$. All variables, related to the rest frame, are labeled below by prime. Evidently, $\langle 1' \rangle = 1$ and $\langle v'_k \rangle = 0$, where $\langle F \rangle = \frac{1}{n}\int F f_e d^3u$.

It is also necessary to define the momentum,

$$nm\langle u'_k \rangle = \frac{1}{c^2} nmc^2 \langle (\gamma' - 1)v'_k \rangle = \frac{1}{c^2} q_k, \tag{4}$$

where q_k is the heat flux. Note that the link between $\langle u'_k \rangle$ and q_k is a purely relativistic effect.

Relation between the thermal energy and temperature is convenient to represent as [3]:

$$W \equiv nmc^2 \langle \gamma' - 1 \rangle = \left(\frac{3}{2} + R\right) nT \quad \text{with} \quad R = \mu\left(\frac{K_3(\mu)}{K_2(\mu)} - 1\right) - \frac{5}{2} = \frac{15}{8\mu} + O\left(\frac{1}{\mu^2}\right). \tag{5}$$

Non-relativistic limit in Eq. (5) is evident.

The next required moment is the flux of momentum, $nm\langle v'_k u'_j \rangle = p\delta_{kj} + \pi_{kj}$, which, similarly to the non-relativistic representation, decomposes into the hydrostatic scalar pressure, $p = nT$, and the (traceless) stress viscous tensor π_{kj} [7],

$$p = \frac{1}{3} nm \langle \frac{u'^2}{\gamma'} \rangle = nT, \qquad \text{and} \qquad \pi_{kj} = nm \langle v'_k u'_j \rangle - p_e \delta_{kj}. \tag{6}$$

The moments related to the collision operator are also required. Since the laws of conservation of momentum and energy in Coulomb collisions of electrons with themselves are automatically satisfied, only the contributions from electron-ion collisions are survived. The electron-ion collisional friction force, R_k^{ei},

$$R_k^{ei} = \int m u'_k C'_{ei} d^3u'. \tag{7}$$

The collision energy exchange rate between the electrons and the classical ions, P^{ei}, can be written as [2]

$$P^{ei} = \int mc^2 (\gamma' - 1) C'_{ei} d^3u' = C_{MJ}(\mu)\left(1 + \frac{2}{\mu} + \frac{2}{\mu^2}\right) P^{ei}_{(cl)}, \tag{8}$$

where $P^{ei}_{(cl)}$ is the classical (non-relativistic) electron-ion energy exchange rate, $P^{ei}_{(cl)} \propto -\frac{T_e - T_i}{T_e^{3/2}}$.

For integration of Eq. (1) with appropriate weights, the weakly relativistic Lorentz transformation of variables from the local frame to the rest frame has to be performed,

$$u_k \simeq u'_k + \gamma_0 \gamma' V_k + \frac{V_k V_j}{2c^2} u'_j, \qquad \gamma \simeq \gamma_0 \gamma' + \frac{V_j u'_j}{c^2}. \tag{9}$$

Apart from that, it is necessary to take into account the invariance of $d^3u/\gamma = d^3u'/\gamma'$.

Performing direct integration of Eq. (1), the standard continuity equation can be obtained,

$$\frac{\partial}{\partial t}(\gamma_0 n) + \frac{\partial}{\partial x_k}(\gamma_0 n V_k) = 0. \tag{10}$$

Formally this equation has exactly the same form as in the fully relativistic approach. A weakly relativistic expansion for γ_0 is assumed, but does not apply here for compactness.

The next is the momentum balance equation, obtained by integration with the weight nmu_k,

$$\frac{\partial}{\partial t}\Big(nm\big(V_k + \delta U_k\big)\Big) + \frac{\partial}{\partial x_j}\Big(\Pi_{kj} + \delta\Pi_{kj}\Big) = enE_k + \frac{1}{c}\big[\mathbf{J}\times\mathbf{B}\big]_k + \big(R_k^{ei} + \delta R_k^{ei}\big). \tag{11}$$

Here and below, the (weakly) relativistic corrections, which disappear at $V/c \to 0$, are labeled by δ. The correction for the momentum is

$$\delta U_k \simeq \frac{1}{nmc^2}\Big(q_k + \big(W+p\big)V_k\Big) + \frac{1}{c^2}\pi_{kj}V_j. \tag{12}$$

The stress tensor is classical-like, $\Pi_{kj} = p\delta_{kj} + \pi_{kj} + nmV_kV_j$, while its correction is

$$\delta\Pi_{kj} \simeq \frac{V_kV_j}{c^2}\Big(W+p\Big) + \frac{1}{c^2}(q_jV_k + q_kV_j) + \frac{V_s}{2c^2}\big(\pi_{ks}V_j + \pi_{js}V_k\big). \tag{13}$$

Correction for friction force (see also Eq. (7)) is

$$\delta R_k^{ei} \simeq \frac{V_k}{c^2}\big(P^{ei} + \frac{1}{2}V_jR_j^{ei}\big). \tag{14}$$

The electric current is $\mathbf{J} = en\mathbf{V} = en(\mathbf{V}_e - \mathbf{V}_i)$. Note that calculating the corrections does not require additional integration.

Integrating Eq. (1) with the weight $nmc^2(\gamma - 1)$, we obtain the energy balance equation,

$$\frac{\partial}{\partial t}\Big(W + K + \delta\mathscr{E}\Big) + \frac{\partial}{\partial x_k}\big(Q_k + \delta Q_k\big) = J_kE_k + R_k^{ei}V_k + \big(P^{ei} + \delta P^{ei}\big). \tag{15}$$

Here, W is defined by Eq. (5), $K = n\frac{mV^2}{2}$, and relativistic correction for total energy is

$$\delta\mathscr{E} \simeq \frac{V^2}{c^2}\Big(W+p\Big) + \frac{1}{c^2}(\pi_{ij}V_i + q_j)V_j. \tag{16}$$

The energy flux is also classical-like,

$$Q_k = \Big(W + p + K\Big)V_k + q_k + \pi_{kj}V_j, \tag{17}$$

while its relativistic correction is

$$\delta Q_k \simeq \frac{V^2}{c^2}\Big(\Big(W + \frac{p}{2}\Big)V_k + \frac{1}{2}q_k\Big) + \frac{V_jV_k}{2c^2}\Big(3q_j + \pi_{jl}V_l\Big). \tag{18}$$

Correction for the electron-ion collision energy exchange rate is equal to

$$\delta P^{ei} \simeq \frac{V^2}{2c^2}P^{ei}. \tag{19}$$

To summarize, we obtained a set of relativistic transport equations, Eq. (10), Eq. (11) and Eq. (15). The terms without the delta are written in the classical-like form and in the non-relativistic limit coincide with the corresponding terms in the classical Braginskii equations, while the correction terms (with the delta) are purely relativistic.

Solution of the linearized relativistic kinetic equations

The standard method for a closure of the transport equations is to solve the linearized kinetic equation by expansion in Sonine polynomials, also called Laguerre polynomials of order $3/2$, i.e. $L_n^{(3/2)}(x)$ with $x = mv^2/2T_e$ (see [1, 7] and the references therein). In the classical case this method is the most effective, but in the relativistic approach its efficiency is questionable because the Sonine polynomials are not eigenfunctions of the RHS of linearized relativistic kinetic equation. As a consequence, a convergence of the series degrades with increasing T_e.

Here, instead of Sonine polynomials, we propose to use generalized Laguerre polynomials $L_n^{(\alpha)}(\kappa)$ of order $\alpha = 3/2 + R(\mu)$, where $\kappa = \mu(\gamma-1)$ and $\mu = mc^2/T$ (see Eq. (5)). An advantage of this method is that $L_n^{(\alpha)}(\kappa)$ are eigenfunctions for the RHS of linearized relativistic equation for arbitrary temperature. For simplicity, we demonstrate the method for $V = 0$. In this case the linearized relativistic kinetic equation can be written as follows,

$$\frac{e}{mc}[\mathbf{v}\times\mathbf{B}]_k \frac{\partial f_{e1}}{\partial u_k} - C_e^{lin}(f_{e1}) = -v_k\left(A_k^{(1)} + \left(\kappa - \frac{5}{2} - R\right)A_k^{(2)}\right)f_{e0}, \tag{20}$$

where $A_k^{(1,2)}$ are the thermodynamic forces,

$$\mathbf{A}^{(1)} = \nabla \log p + \frac{e\mathbf{E}}{T} \quad \text{and} \quad \mathbf{A}^{(2)} = \nabla \log T. \tag{21}$$

While the angular dependence can be found, as usual, by expansion in spherical Legendre harmonics, the energy dependence of f_{e1} can be represented as the series in $L_n^{(\alpha)}(\kappa)$. Since $L_0^{(\alpha)}(\kappa) = 1$ and $L_1^{(\alpha)}(\kappa) = \frac{5}{2} - \kappa - R$, the RHS of Eq. (20) can be represented by the linear combination of only these terms. As a consequence, the series for the solution converges as fast as in the classical case, but with validity for arbitrary temperature.

Following this line, all transport coefficients are expressed by integrals as follows,

$$\begin{aligned} M_{nm}^{ab} &= \tfrac{\tau_{ab}}{n_a} C_{MJ}^{-1} \int \mathrm{d}^3u\, u_k L_n^{(\alpha)}(\kappa) C^{ab}\left(w\, \tfrac{m_a u_k}{T_a} L_m^{(\alpha)}(\kappa) f_{a0};\, f_{b0}\right), \\ N_{nm}^{aa} &= \tfrac{\tau_{aa}}{n_a} C_{MJ}^{-1} \int \mathrm{d}^3u\, u_k L_n^{(\alpha)}(\kappa) C^{aa}\left(f_{a0};\, w\, \tfrac{m_a u_k}{T_a} L_m^{(\alpha)}(\kappa) f_{a0}\right), \end{aligned} \tag{22}$$

where $w = \kappa^R\left(\frac{2}{\gamma+1}\right)$; index a is for e and b for e,i. Formally, these coefficients have the same structure as in the classical case, and in the non-relativistic limit they coincide.

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