Beyond resonance broadening and quasilinear theory: towards Kubo >1

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The behavior of charged particles in an electric field that includes multiple waves can be broadly classified into two categories. In the first scenario, where the amplitude of the waves is small, each particle interacts with only one wave in a linear manner, resulting in regular dynamics. However, as the amplitude of the waves increases, particles begin to interact with multiple waves, leading to chaotic dynamics. From the perspective of the waves, these non-linear interactions between waves and particles can cause energy to cascade from one wave to many others, as described by Kolmogorov in 1941 [1]. This results in the generation of lower and higher modes, with modes referring to the system's eigenfunctions. In a state of fully developed turbulence, the electric field modes can be in random phases with each other, making the dynamics complex. Particles can be trapped in a wave and then released due to the interaction with a different mode. In a plasma without collisions, turbulence is a primary cause of particle transport and energy loss through plasma wave-particle resonances. Furthermore, turbulence significantly contributes to plasma heating through particle acceleration [2], which has significant implications in various plasma phenomena such as instabilities driven by energetic particles, magnetic reconnection in space plasmas [3], stimulated Raman scattering [4], and laser-plasma interactions [5].

The study of turbulence gained momentum in the early 1960s with the advent of quasi-linear theory [6] and resonance broadening [7]. The initial goal of quasi-linear theory was to investigate non-equilibrium plasma dynamics by disregarding mode coupling, considering only some non-linear terms, and focusing on the temporal evolution of particle distribution. It was assumed that turbulence does not trap particles. Under these assumptions, it's possible to derive an expression for transport as an expansion of the electric field amplitude. However, for moderate amplitudes (or low dispersion), non-linear terms can't be ignored, leading to a broadening of wave-particle resonances and effects of mode coupling. Incorporating the effect of quasi-linear diffusion in the model of particle motion, known as re-normalization, allows for accounting for this broadening.

Studies have shown strong qualitative and quantitative agreement with quasi-linear theory

[8, 9] at low electric field amplitudes for electric fields with random phases. In this study, we explore the effects of high amplitude turbulence [10]. When considering the self-consistent problem, which accounts for the modification of mean fields, it has been experimentally and numerically demonstrated that re-normalization is not necessary at low amplitudes. However, recent numerical simulations for the self-consistent bump-on-tail instability reveal that the quasi-linear theory fails to predict plasma processes at low amplitude, enhanced diffusivity, and phase-space restructuring.

One key parameter that determines the applicability of quasi-linear and resonance-broadening theories is the Kubo number [11]. This parameter is defined as the ratio of the autocorrelation time τ_0, which is the time it takes for the turbulent electric field to change its shape, to the flight or bounce time τ_b, which is the time it takes for a trapped particle to complete an orbit. The Kubo number is therefore defined as:

$$K = \tau_0/\tau_b \ . \tag{1}$$

This expression can be rewritten in terms of the electric field amplitude, resulting in $K \propto E^{1/2}$, since $\tau_b \propto E^{-1/2}$. It's important to note that for the application of quasi-linear and resonance broadening theories, the Kubo number should be significantly less than one ($K \ll 1$) [6].

The Kubo number plays a crucial role in turbulence-related literature. Firstly, it helps differentiate the type of trajectory performed. For instance, for $K \ll 1$, one can visualize particles hopping between arcs of trapped trajectories. On the other hand, for $K \geq 1$, particles perform multiple trapped orbits separated by small jumps between two different trapped particle trajectories. Furthermore, the Kubo number appears in various plasma turbulence theories, such as quasi-linear and mean-field theories [12], with the latter describing the relaxation transport in plasmas.

Calculations based on mixing length theory assume that the Kubo number is approximately equal to one ($K \simeq 1$) [13], which contradicts the requirement of $K \ll 1$ for the validity of the quasi-linear theory. Despite this discrepancy, both mixing-length and quasi-linear theories are often used concurrently. This study aims to investigate the statistical diffusion coefficient of test particles in a prescribed one-dimensional turbulent electric field. We will compare results from numerical trajectories against quasi-linear theory, including resonance broadening. We will examine diffusion for ion-acoustic and Langmuir dispersion relations to compare the effects of dispersivity, and a Gaussian amplitude electric potential spectrum will be used. We will investigate different regimes of particle trapping [8, 9], particularly for $K \geq 1$. We aim to answer the key questions: To what extent does quasi-linear theory work outside the $K \ll 1$ regime? Is there a way to expand, correct, or replace quasi-linear theory to describe a plasma in the $K > 1$ regime? In this work, we are particularly interested in many resonances.

Turbulent electric field: Case of Gaussian spectrum

We focus on the dynamics of test particles in a predetermined turbulent electric field. We've developed an algorithm that calculates the trajectories of N particles using a fourth-order Runge-Kutta method in the given electric field. At the start ($t = 0$), N test particles are initialized with random velocities and positions. Notably, particle velocities are distributed in a narrow Gaussian probability around a mean velocity v_0, while particle positions are uniformly distributed within one periodicity interval of the electric field $[0;L[$. At every time step, we compute trajectory diagnostics such as particle distributions, statistical moments, and the maximum finite-time Lyapunov exponents.

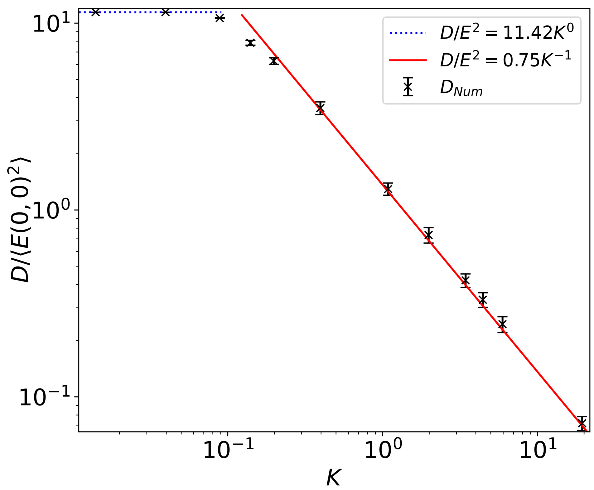

Figure 1: *Numerical diffusion coefficient in crosses as a function of the Kubo number K, for* $v_0 = 1.95$. *The power law fit dependence on Kubo number of diffusion is in solid and dashed lines.*

Our first investigation centered on the dependence of the diffusion coefficient on the Kubo number at a fixed initial particle velocity. For times on the order of τ_0, in figure 1 we observed three distinct regimes in the normalized diffusion coefficient from numerical simulations as a function of the Kubo number. For $K \ll 1$, the normalized diffusion remains constant, as predicted by quasi-linear theory. This is followed by a transition regime where the Kubo number is a few percent, and here the diffusion coefficient decreases non-linearly, as observed and predicted by resonance broadening [8, 9]. For $K > 0.5$, the normalized diffusion evolves as a power of the Kubo number, K^{-1}. Similar results were found for the ion-acoustic dispersion relation.

Conclusion

To summarize, our study delved into the diffusion of charged particles in a predetermined one-dimensional turbulent electric field using numerical simulations and quasi-linear theory. We measured statistical diffusion coefficients at various Kubo number values, utilizing a Gaussian amplitude spectrum and realistic plasma dispersion relations: Langmuir and ion-acoustic dispersions. Initially, we examined diffusion at a low Kubo number as a function of the initial particle velocity. The results from our numerical simulations were in line, both qualitatively and quantitatively, with quasi-linear theory, including resonance broadening. These results also aligned with previous studies that explored diffusion at low Kubo numbers and for non-physical dispersion relations [8, 9]. Furthermore, we extensively studied diffusion coefficients outside the quasi-linear regime, at large Kubo numbers, as a function of the Kubo number. Here, we found that diffusion scales as a power law, $K^3 \propto E^{3/2}$, which we interpret as a random walk diffusion of the centers of trapped-particle trajectories in the velocity direction. Lastly, we investigated the diffusion generated by phase-space structures and their impact on particle dynamics. In conclusion, quasi-linear and resonance-broadening theories accurately predict particle diffusion in the small Kubo numbers limit ($K < 10\%$) for realistic plasma dispersion relations and a predetermined turbulent electric field. A simple proportionality of diffusion is measured ($D \propto E^{3/2}$) and predicts the evolution for large Kubo numbers ($K \gg 1$). Additionally, we demonstrated that phase-space structures significantly dominate the diffusion of particles in plasmas with low-amplitude turbulence. However, more work is needed to enhance our understanding of turbulence, transport, and diffusion.

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