Rearrangement and Free Energy for Understanding Plasma Stability

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Introduction

For a wide variety of plasma systems, it is helpful to quantify the kinetic energy that can be extracted from a given phase-space configuration: the “free” or “available” energy. For example, in alpha channeling [1], the goal is to extract energy from fusion products using wave-particle interactions, and the extracted energy will not exceed the theoretically available value. The available energy can also measure how much energy could be accessible to feed instabilities.

There are a number of different ways of quantifying available energy. Each corresponds to a different rule for how the distribution of particles may be rearranged in phase space. The Gardner free energy is the energy that can be released through any process that conserves the volumes of phase space elements [2, 3, 4]. The diffusively accessible free energy is the energy that can be released by those processes that mix the volumes of phase space elements [5, 6, 7, 8, 9, 10]. In this presentation we will discuss progress on characterizing these different free energies, and possible applications, with a particular focus on characterizing instabilities.

Defining the Free Energies

Phase-space rearrangements can be defined either for discrete or continuous systems. It is often helpful to build intuition with discrete systems, but most applications of interest in plasma physics involve continuous phase space. In the simplest, finite discrete case, one can visualize phase space as a set of “boxes” containing different populations, with each box associated with a different energy. For example, for an $N = 3$ discrete system, one could assign energies $\varepsilon_0 = 0$, $\varepsilon_1 = 1$, and $\varepsilon_2 = 2$ to the respective boxes, so that the configuration with populations 3, 1, and 2 has dimensionless energy content $3\varepsilon_0 + \varepsilon_1 + 2\varepsilon_2 = 5$.

For this case, Gardner restacking leads to a ground state as follows:

$$\boxed{3}\boxed{1}\boxed{2} \rightarrow \boxed{3}\boxed{2}\boxed{1}$$

such that the ground state has energy $2\varepsilon_1 + \varepsilon_2 = 4$. Diffusive exchange, on the other hand, instead minimizes the final energy by taking

$$\boxed{3}\boxed{1}\boxed{2} \rightarrow \boxed{3}\boxed{1.5}\boxed{1.5}$$

for a final energy of 4.5. In general, diffusive exchange operations can map the same initial state to any of a spectrum of final ground states, with an associated spectrum of ground state energies. The lowest-energy accessible ground state, associated with the largest possible release in energy from the initial state, is the diffusively accessible free energy originally proposed by Fisch and Rax [5]. Explicitly calculating this optimal ground state is computationally difficult; Hay, Schiff, and Fisch showed that for the N-state discrete system the computational complexity scales like $\mathscr{O}(N^{N^2})$ [6]. For discrete systems, this diffusively accessively free energy is always less than the release possible through Gardner restacking (except for the degenerate case in which the system starts in a ground state). However, note that the condition for a configuration to be a ground state is the same for both classes of rearrangement; the system is in a ground state if and only if the population of a given box is a monotonically decreasing function of energy.

The intuition for continuous phase space is essentially that it is the limit of the N-state discrete problem as $N \to \infty$, with energies that are very close between adjacent boxes (though, as we will discuss, some aspects of the problem turn out to be fundamentally different in this continuous limit). In the case of Gardner restacking, the continuous analog is sometimes known as the "symmetric decreasing rearrangement" in pure mathematics [11, 12, 13, 14, 15]. For the diffusive case, one can understand the free-energy problem as the minimization of

$$W_{\text{final}} \doteq \lim_{t \to \infty} \int \varepsilon(v) f(v,t) \, \mathrm{d}v \tag{1}$$

with

$$\frac{\partial f}{\partial t} = \int K(v, v', t) \big[f(v', t) - f(v, t) \big] \, \mathrm{d}v', \tag{2}$$

where $\varepsilon(v)$ is the energy per particle and K is a kernel that is positive-semidefinite in the sense that $K(v, v', t) \geq 0$ and symmetric in the sense that $K(v, v', t) = K(v', v, t)$ for all v, v', and t. Then the problem consists essentially of determining the optimal K [5]. More generally, the spectrum of accessible ground states is the set of configurations f that can be reached by kernels K satisfying these two requirements. The work discussed in this contribution focuses on progress in the characterization of this spectrum.

For both the Gardner [3, 16] and diffusive [8] free energies, it is possible to impose additional conservation laws (for example, to enforce adiabatic invariants). There is evidence [4] that these conservation laws make it possible to use free energy calculations to make practical predictions for a wider range of real systems. The remainder of this contribution will not consider these additional conservation laws. However, the generalization of the ideas discussed here to those free energy formulations is relatively straightforward.

Maximum-Energy Diffusively Accessible Ground States

Most of the literature on the diffusive exchange problem focuses on determining that maximum possible energy extraction that can be achieved through diffusive operations – that is, the lowest-energy accessible ground state. Much of the early work was motivated by problems like alpha-channeling, where the objective is to remove as much energy as possible from a population of particles; for these applications, the minimum-energy ground state is the object of greatest interest. However, there is growing evidence to suggest that measures of free energy have applications to the behavior of "naturally occurring" (typically deleterious) instabilities [4]. For these applications, it is desirable to characterize the whole spectrum of possible ground states. Thus, it is interesting to consider the *highest*-energy diffusively accessible ground state, corresponding to the smallest possible release of energy that stabilizes a distribution [10].

When considering the maximum-energy accessible ground state, we include an additional rule that a mixing operation is allowed only if it releases energy. This aligns with our intuitions for real instabilities. This rule does not change the minimum-energy accessible state, but if it is not included, the maximum-energy state is that in which all states have the same final population (often leading to a final energy greater than what the system started with).

It is possible to calculate the maximum-energy diffusively accessible ground state in certain cases. These include the $N = 3$ discrete system for arbitrary initial conditions. They also include the continuous bump-on-tail distribution. It turns out that the quasilinear plateau solution – in which a region surrounding the bump is flattened and the distribution is otherwise left unchanged – is in fact the maximum-energy diffusively accessible ground state in this case [10].

Minimum-Energy Diffusively Accessible Ground States

The lowest-energy diffusively accessible state for continuous systems is defined by Eqs. (1) and (2). Surprisingly enough, it is possible to show that the maximum energy that can be liberated from a system using diffusive operations is exactly equal to the energy released by Gardner restacking, and the minimum-energy accessible ground state is the same as the Gardner ground state. This is counterintuitive, given that diffusive exchange operations create entropy while Gardner restacking operations do not. However, it turns out that sequences of mixing operations can lead to a ground state while producing vanishingly little entropy, if they are picked carefully. This is described in greater detail in Ref. [9], but the proof is constructive; it is possible to write down a prescription for a sequence of mixing operations that reproduces Gardner restacking in the appropriate limit.

Conclusion

The concept of free or available energy provides us with a very general set of tools with which to understand limits on the behavior of different kinds of phase space rearrangements. This is helpful for applications involving wave-particle interactions (like alpha-channeling) as well as instabilities. The work discussed in this contribution focuses on characterizing the upper and lower bounds of the energy that can be released using diffusive phase-space mixing operations. Ongoing work includes applications to instabilities in mirror-type configurations.

Acknowledgments

This work was supported by Grant No. DOE DE-SC0016072, and by the U.S. DOE Fusion Energy Sciences Postdoctoral Research Program administered under contract DE-SC0014664.

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