Hybrid, tokamak-pertinent equilibria with toroidal plasma rotation

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Motivation and Aim

Experimental and astrophysical plasmas usually exhibit complex multi-scale behaviour featuring both fluid bulk and energetic particle populations. Such plasmas cannot be described by fluid models which are associated with Maxwellian distribution functions. On the other side, employment of Vlasov equation in the presence of axisymmetry leads to equilibria with purely toroidal currents in view of the two particle constants of motion, i.e. the energy and the toroidal angular momentum. To overcome these difficulties one can appeal to hybrid fluid-kinetic models.

The aim of the present study is to construct hybrid equilibrium states for plasmas with kinetic ions and massless fluid electrons, assuming isothermal electrons and deformed Maxwellian distribution functions for the kinetic ions

The Hybrid model

The dimensionless hybrid model consists of a Vlasov equation for kinetic ions, a generalized Ohm's law derived from the electron momentum equation, the Maxwell equations, and an equation of state for the massless fluid electrons:

$$\mathbf{v}\cdot\nabla f + d_i^{-2}\left(\mathbf{E}+\mathbf{v}\times\mathbf{B}\right)\cdot\nabla_v f = 0 \tag{1}$$

$$-\nabla\Phi = \frac{1}{n_e}\left[(\nabla\times\mathbf{B}-\mathbf{J}_k) - d_i^2\nabla P_e\right], \quad \mathbf{E}=-\nabla\Phi \tag{2}$$

$$d_i^2\beta_A^2\nabla\cdot\mathbf{E} = (n-n_e), \quad \nabla\times\mathbf{B}=\mathbf{J}, \quad \nabla\cdot\mathbf{B}=0 \tag{3}$$

$$P_e = \kappa n_e\,. \tag{4}$$

Here, $v_A = B_0/(\mu_0 m n_0)^{1/2}$ and $d_i = (1/R_0)\left(m/(\mu_0 n_0 e^2)\right)^{1/2}$ are reference Alfvén velocity and ion skin depth, respectively; $n = (1/n_0)\int d^3v f(\mathbf{x},\mathbf{v},t)$ is the ion density; $\mathbf{J} = \mathbf{J}_e + \mathbf{J}_k$ is the total current density as sum of the electron and ion contributions with $\mathbf{J}_k = (n_0 d_i v_A)^{-1}\int d^3v\mathbf{v}f$; $\kappa := k_B T_{e0}/(d_i^2 m v_A^2)$; $\beta_A^2 = v_A^2/c^2$; and R_0, B_0, n_0, T_{e0} are constant reference quantities to be defined later. Cold electrons correspond to $\kappa = 0$ and thermal electrons to $\kappa \neq 0$. In the limit $\beta_A^2 \to 0$ we obtain the quasineutrality condition $n_e = n$.

Axisymmetric equilibrium formulation

In cylindrical coordinates (r,ϕ,z) the axisymmetric magnetic field and current density can be expressed in terms of the scalar quantities $\psi(r,z)$ and $I(r,z)$ as

$$\mathbf{B} = I\nabla\phi + \nabla\psi(r,z)\times\nabla\phi, \quad \mathbf{J} = \nabla\times\mathbf{B} = -\Delta^*\psi\nabla\phi + \nabla I\times\nabla\phi\,, \tag{5}$$

where $\Delta^* := r\partial/\partial r\left((1/r)\partial/\partial r\right) + \partial^2/\partial z^2$ is the elliptic Shafranov operator. The poloidal magnetic flux-function $\psi(r,z)$ labels the magnetic surfaces.

According to Jeans's theorem, distribution functions of the form $f = f(C_1, C_2, ...)$, where C_i are particle constants of motion, are solutions to the Vlasov equation. For an axisymmetric system constants of motion are the energy and the toroidal angular momentum:

$$\tilde{H} = \frac{v^2}{2} + d_i^{-2}\Phi\,, \quad \tilde{H} = \frac{H}{d_i^2 m v_A^2} \tag{6}$$

$$\tilde{p}_\phi = r v_\phi + r A_\phi = r v_\phi + d_i^{-2}\psi\,, \tag{7}$$

where A_ϕ is the ϕ-component of the vector potential. We will further consider distribution functions of the form

$$f = f(\tilde{H}, \tilde{p}_\phi) = e^{-\tilde{H}}\, g(\tilde{p}_\phi)\,, \tag{8}$$

with the function $g(\tilde{p}_\phi)$ to be determined. The choice (8) implies that the (kinetic) ion current density is purely toroidal:

$$\mathbf{J}_k = r J_{k\phi} \nabla\phi\,, \;\; J_{k\phi} = \int v_\phi\, f\, d^3v.$$

Projecting the Ohm's law (2) along $\mathbf{B}$ and the toroidal direction $\nabla\phi$, respectively, yields:

$$n = exp\left[\frac{\Phi - G(\psi)}{d_i^2 \kappa}\right]\,, \quad I = I(\psi)\,, \tag{9}$$

where $G(\psi)$ is a free surface function. For $G(\psi) = \text{const.}$ we recover the Boltzmann distribution.

The $\nabla\psi$ projection of Ohm's law (2) furnishes

$$\Delta^*\psi + I I'(\psi) + r^2 \mathscr{Z}(r,\psi) = 0\,, \tag{10}$$

$$\mathscr{Z}(r,\psi) := \frac{1}{r}\overbrace{\int d^3v\, v_\phi f}^{J_{k\phi}} - G'(\psi)\overbrace{\int d^3v\, f}^{n}\,. \tag{11}$$

For distribution functions of the form (8) the integrals in (11) for $J_{k\phi}$ and n can be expressed in terms of the two arbitrary functions $G(\psi)$ and $g(\tilde{p}_\phi)$. Henceforth the tilde, $\sim$, will be dropped.

We can show that the function $\mathscr{Z}(r,\psi)$ can be derived by a "pseudopotential" function $V(r,\psi)$ as $\mathscr{Z} = \partial V/\partial\psi$, where

$$V(\psi, r) = d_i^2(\kappa+1)\left[\frac{2\pi e^{-G(\psi)/d_i^2}}{r}\int_{-\infty}^{+\infty} dp_\phi\, e^{-\frac{(p_\phi - \psi/d_i^2)^2}{2r^2}}\, g(p_\phi)\right]^{\frac{1}{\kappa+1}}. \tag{12}$$

Consequently, the generalized Grad-Shafranov (GGS) equation (10) can be written in the form

$$\Delta^*\psi + I I'(\psi) + r^2\frac{\partial V}{\partial\psi} = 0\,. \tag{13}$$

To solve Eq. (13) we can specify $V(\psi, r)$ to be a known mathematical function or it can be inferred from experimental data. Knowing V enables the solution of the partial differential equation (13) under appropriate boundary conditions to determine ψ. The integral of equation (12) can be determined by expressing the function $g(p_\phi)$ in terms of Hermite polynomials as shown in the Appendix.

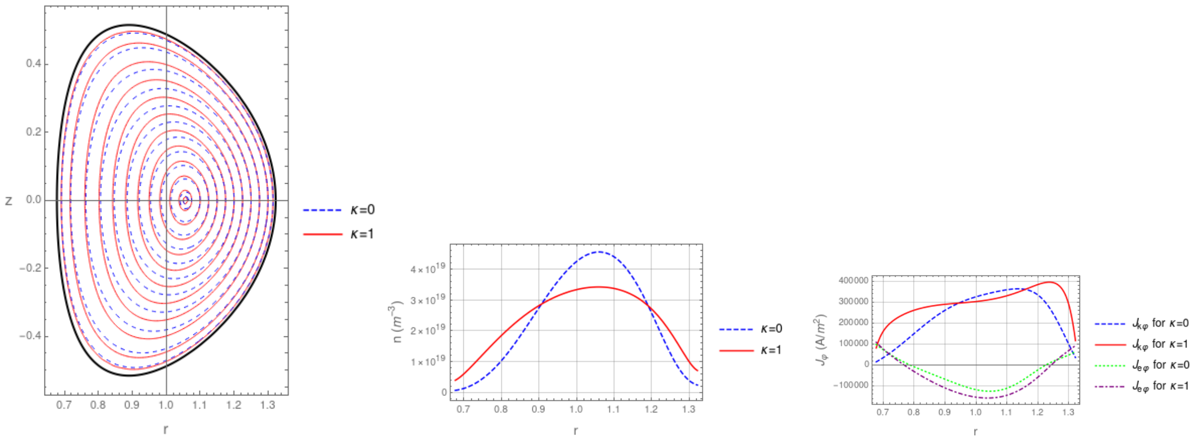

Figure 1: Magnetic surfaces of an equilibrium with cold electrons (blue dashed lines) and an equilibrium with thermal electrons (red solid lines) (left panel). Particle density profiles for $\kappa = 0$ and $\kappa = 1$ (medium panel), and the fluid-electron and kinetic-ion contributions to the current density for $\kappa = 0$ and $\kappa = 1$ (right panel).

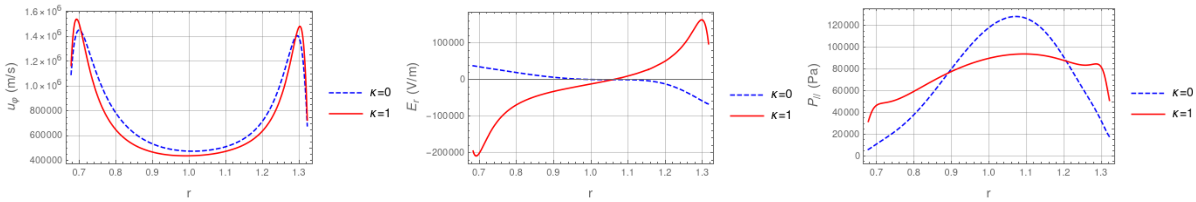

Figure 2: Profiles of the toroidal rotation velocity (left panel), electric field (medium panel) and pressure-tensor component parallel to **B** (right panel) *exhibiting H-mode characteristics.*

Tokamak equilibria

We make that following ansatz for the free functions:

$$I(\psi) = (I_0 + I_1\psi + I_2\psi^2)e^{-(\psi-\psi_a)^2/\eta}\,, \quad G(\psi) = \alpha(\psi-\psi_a)^2\,,$$

where ψ_a is the value of ψ on the magnetic axis. Expanding the function

$$\left\{V/\left[d_i^2(\kappa+1)\right]\right\}^{\kappa+1}\exp\left[G(\psi)/d_i^2\right]$$

in power series up to second-order in ψ and following the procedure described in the Appendix we find

$$V = (\kappa+1)d_i^2\left\{e^{-G(\psi)/d_i^2}\left[V_0(r)+V_1\psi+V_2\psi^2\right]\right\}^{1/(\kappa+1)}\,,$$

$$f(H,p_\phi) = \left[c_0+\sqrt{2}c_1 p_\phi + c_2(2p_\phi^2-2)\right]e^{-H}\,.$$

Then, the GGS equation (13) is solved numerically by the Finite Elements Method for a fixed D-shaped boundary cross-section, on which $\psi|_{\partial\mathscr{D}} = 0$, using the following ITER parametric values: aspect ratio $\varepsilon = 0.32$, triangularity $\delta = 0.34$, elongation $k = 1.6$, major radius $R_0 = 6.2\,m$, central magnetic field $B_0 = 5\,T$. Also, $n_0 = 2.1\times10^{19}$ m^{-3}. Two cases have been considered: cold electrons (κ=0) and hot electrons (κ=1). The results are illustrated in Figures 1, 2 and 3.

Conclusions

By employing a hybrid fluid-Vlasov equilibrium model, featuring massless, isothermal fluid electrons and kinetic ions, we derived a generalized Grad-Shafranov equation. Upon solving