Scaling laws of the plasma velocity in visco-resistive magnetohydrodynamic systems

A. Krupka[1], M.-C. Firpo[1]

[1] *Laboratoire de Physique des Plasmas (LPP), CNRS, Sorbonne Université, École polytechnique, Institut Polytechnique de Paris, 91120 Palaiseau, France*

Abstract

We consider a visco-resistive magnetohydrodynamic modeling of a steady-state incompressible tokamak plasma with a prescribed toroidal current drive, featuring constant resistivity η and viscosity ν. We reintroduce in the traditional Grad-Shafranov equation the dissipative viscous term and the non-linear $(\mathbf{v} \cdot \nabla)\mathbf{v}$ term coming from the steady-state Navier-Stokes equation [1, 2, 3, 4, 5, 6]. It is shown that the plasma velocity root-mean-square behaves as $\eta f(H)$ as long as the inertial term remains negligible, where H stands for the Hartmann number $H \equiv (\eta\nu)^{-1/2}$, and that $f(H)$ exhibits power-law behaviours in the limits $H \ll 1$ and $H \gg 1$. In the latter limit, we establish that $f(H)$ scales as $H^{1/4}$, which is consistent with numerical results. These use the finite element method through the open-source platform FreeFem++ for solving partial differential equations [7].

Behaviour of the plasma velocity

The computation of visco-resistive axisymmetric steady states involves solving the steady-state incompressible Navier–Stokes equation (1)-(2) along with the solenoidal condition (3), Faraday's law (4), Ampère law (5) and Ohm's law (6)

$$(\boldsymbol{v} \cdot \nabla)\boldsymbol{v} = \boldsymbol{J} \times \boldsymbol{B} - \nabla p + \nu \nabla^2 \boldsymbol{v}, \tag{1}$$

$$\nabla \cdot \boldsymbol{v} = 0, \tag{2}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{3}$$

$$\nabla \times \boldsymbol{E} = 0, \tag{4}$$

$$\nabla \times \boldsymbol{B} = \boldsymbol{J}, \tag{5}$$

$$\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B} = \eta \boldsymbol{J}. \tag{6}$$

on a tokamak poloidal plasma cross-section Ω. Let E_0 be the toroidal electric field. In the present study, we are interested in the behavior of the system for a given ratio of E_0/η. This quantity may be viewed as the only explicit drive appearing in the dimensionless system of equations. Combining the Navier-Stokes equation and Ohm's law yields

$$\boldsymbol{\omega} \times \mathbf{v} = -\nabla p^* + \left(\frac{\mathbf{E}}{\eta} + \eta^{-1} \mathbf{v} \times \mathbf{b} \right) \times \mathbf{b} + \nu \nabla^2 \mathbf{v}. \tag{7}$$

Using the Hartmann number, H, defined as $H = (\eta\nu)^{-1/2}$, this reads and introducing $\mathbf{v} = (\eta/\nu)^{1/2}\, \widetilde{\mathbf{v}}$:

$$\eta^2 H^2 \widetilde{\boldsymbol{\omega}} \times \widetilde{\mathbf{v}} = -\nabla p^* + \left(\frac{\mathbf{E}}{\eta} + H \widetilde{\mathbf{v}} \times \mathbf{b} \right) \times \mathbf{b} + H^{-1} \nabla^2 \widetilde{\mathbf{v}}. \tag{8}$$

If we neglect inertial term and eliminate pressure by taking the curl of the equation then we are left with

$$\nabla \times \left[\left(\frac{\mathbf{E}}{\eta} + H \widetilde{\mathbf{v}} \times \mathbf{b} \right) \times \mathbf{b} + H^{-1} \nabla^2 \widetilde{\mathbf{v}} \right] = \mathbf{0} \tag{9}$$

and, at given $\mathbf{E}/\eta$, plasma velocity $\widetilde{\mathbf{v}}$ must be a function of H only, meaning that $\mathbf{v}/\eta = H\widetilde{\mathbf{v}}$ is a function of H only. This prevalence of the Hartmann number was already inferred in the Reverse Field Pinch [8] where visco-resistive MHD simulations showed that the transition from multiple to quasi-single helicity states is controlled by the Hartmann number.

Numerical results

Fig. 1 on the left depicts the computed steady-state toroidal velocity fields for two different Hartmann numbers: $H = 10$ and $H = 10^5$ at a given resistivity η.

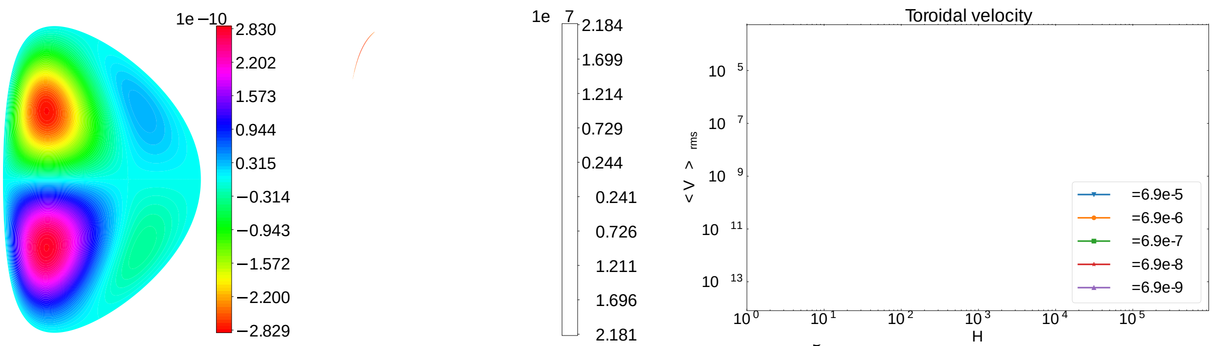

Figure 1: (On the left) Toroidal velocity field for $H = 10$ and $H = 10^5$ with η=6.9e-9 and $E_0/\eta = 0.43$. (On the right) Root-mean square of the toroidal velocity field in Alfvén velocity units as a function of the Hartmann number in log-log scale for different values of the resistivity with $E_0/\eta = 0.43$.

The log-log plot of Fig. 1 on the right demonstrate that, at a given E_0/η ratio, the velocity is proportional to the resistivity and to some function depending solely on the Hartmann number.

Scaling Laws

The scaling of velocity in the first regime where $H \ll 1$ was predicted analytically [2]. The toroidal velocity in this limit scales with H^4 while the poloidal velocity scales with H^2. We analytically predict the velocity behaviour in $H \gg 1$ regime by considering the boundary layer equations [9]. This gives the boundary layer thickness scaling as $\delta \sim \frac{1}{\sqrt{H}}$. Consequently, $< \widetilde{v}_\phi >_{rms} \sim\ H/\eta\delta^{-1/2}$, so that in the original velocity variable and using boundary layer