# Non-axisymmetric generalisation of an instability criterion for disks with suprathermal toroidal fields

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**Abstract**

In upcoming work, we discuss the visual aspects of non-axisymmetric instabilities in thin accretion disks, which correspond to weak field and large plasma-beta equilibrium states. These modes make up an unstable 2D continuum and are all corotating at the local Doppler frequency and confined between two resonances with the Alfvén continua. We here focus on non-axisymmetric modes in disks with suprathermal toroidal fields, and in addition propose that stability criteria that were derived in earlier work for axisymmetric perturbations generalise to non-axisymmetric perturbations. In these regimes, even for axisymmetric modes, rotation and the Doppler-Coriolis shift produce intricate MHD spectra where the usual ordering of discrete modes is locally not satisfied.

## Introduction

Accretion disks are assumed to be turbulent as a result of linear instabilities, most notably the axisymmetric magneto-rotational instability. Over recent years, the importance of non-axisymmetric modes has become clear, since they provide necessary conditions for dynamo activity and can live in regimes where the MRI is typically quenched [1]. In thin, weakly magnetised disks, recent progress includes a new type of instability, the Super-Alfvénic Rotational Instability (SARI). Contrary to earlier cylindrical (global) treatments producing discrete unstable modes whose eigenfunctions cluster to the inner or outer disk radius, these SARIs form wave packages that are global in nature but nevertheless localised between resonances with the Alfvén continua, producing skin currents that act as virtual walls. However, non-axisymmetric instabilities in disks with strong suprathermal magnetic fields are largely unexplored. For $m = 0$ modes, [2] revisited the Cartesian analysis by [3] of strong, suprathermal toroidal fields with a seed vertical field in a cylindrical model. They confirmed that at low values of plasma-beta, two new instabilities (slow and a hybrid one) appear that satisfy the local instability criterion. We revisit these results to account for $m \neq 0$ instabilities.

## Methods

We perform an MHD spectrosopic study using the `Legolas` code[1], which reformulates the linearised MHD equations into weak Galerkin formulation and solves for unknown eigenfre-

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[1]`https://legolas.science`

quencies and eigenfunctions using finite elements. We use a cylindrical model with shear flow and a homogeneous background as in [2]. Small perturbations to the equilibrium are assumed to be normal modes $\hat{f}(r)\exp\left[i\left(m\theta + kz - \omega t\right)\right]$, where $m$ and $k$ are the azimuthal and vertical wavenumbers and $\omega = \omega_r + i\omega_i$ the complex eigenfrequency with growth rate $\omega_i$. Note that in our cylindrical model, the radial variation is not through a radial wavenumber, but through a non-trivial function $\hat{f}(r)$ that quantifies the radial variation into the Fourier coefficient.

We use non-dimensionalised quantities for our study, determined by $\rho = 1$, $B_\theta = 0.4$, $B_z = 0.01$, $c_S = 0.05$ for density, toroidal and poloidal field and sound speed. The unit time is set by our choice of Keplerian rotation frequency $\Omega_K = 1$. The rotational regime is hence slightly super-Alfvénic ($v_A \approx 0.4$) with a suprathermal field ($\beta = 0.03$). The domain is taken to be $r \in [1,2]$ and assumed invariant in the vertical direction. This approximation is valid if the vertical wavelength fits inside the pressure scale height of the disk, $k = \frac{2\pi}{\lambda_z} > \frac{2\pi}{2H} = \frac{\pi}{\sqrt{2}c_S} = 44.5$ as in [2]. We take $\gamma = 1.00001$ to approximate their assumption of isothermal perturbations.

Since the background is assumed homogeneous, pressure gradients do not contribute to the radial force balance, which only contains Keplerian rotation and field curvature: $\Omega^2 = \Omega_K^2 + \frac{B_\theta^2}{r^2}$. This leads to a quasi-Keplerian background flow $\Omega(r) = r^{-3/2}\left(1 + rB_\theta^2\right)^{1/2}$. Another important difference with our earlier work [4, 1] is that, since $B_z$ has no equilibrium variation, the MHD continua collapse to a degenerate point for $m = 0$. Nevertheless, $m \neq 0$ complicates the analysis again, since then the Doppler range $\Omega_0(r) = m\Omega(r)$ is non-zero, providing a range of continuum frequencies $\Omega_{A,S}^{\pm} := \omega_{A,S} \pm \Omega_0$. Here, $\omega_A = \frac{m}{r}B_\theta(r) + kB_z$ and $\omega_S = \sqrt{\frac{\gamma p}{\gamma p + B^2}}\omega_A$ are the Alfvén and slow frequencies for this equilibrium. It is exactly the interaction of these singular ranges that leads to interesting results in thin disks (the SARIs).

[2] confirmed for $m = 0$ that the usual MRI instability criterion $0 < \omega_A^2 < 3\Omega$ is only valid for lower $v_A$, but should in this regime be replaced by the instability criterion

$$(kB_z)^2 < \frac{v_{A\theta}^4}{c_S^2} - 1 =: k_c^2 \tag{1}$$

derived by [3] in the local approximation for strong toroidal fields. In this short work, we establish numerically that the instability criterion is valid for non-axisymmetric instabilities with low $m$, not unlike how the MRI criterion generalises to $m \neq 0$ [4]. We confirm that in the regime discussed here, the MRI is replaced by two truly new instabilities, a hybrid and a slow one, as shown by [2], but with additional branches appearing in the non-axisymmetric case.

**Results**

We first revisit the analysis by [2] for the $m = 0$ case. Figure 1 features the unstable and oscillating part of a spectrum at low wavenumber $k$, where the hybrid and slow instabilities
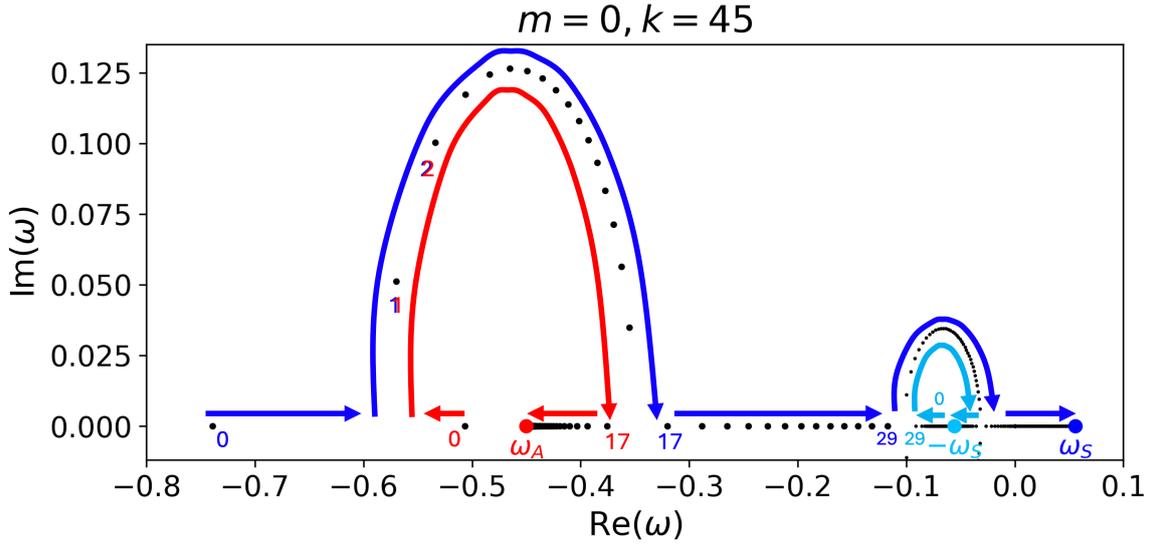
Figure 1: *Spectrum for the MRI in the suprathermal toroidal field shows how the discrete slow modes clustering towards the degenerate continua have to pass through the other continua, which renders them unstable. Two unstable branches appear, with hybrid slow-Alfvén and slow-slow coupling as the sequences merge.*

appear as two sections of the same branch of discrete modes clustering to $\omega_S$ (blue arrow). In this extreme case, the oscillating discrete modes clustering to $\omega_S$ encounter two singularities and avoid them by becoming unstable and merging with the sequence of discrete Alfvén modes (red) and $-\omega_S$ slow modes (light blue), indeed giving them a hybrid slow-Alfvén or slow-slow nature. The (anti-)Sturmian ordering of the eigenvalues in static MHD (where the eigenvalues form a sequence strictly decreasing in frequency where the nodes of subsequent eigenfunctions increase by one – number of nodes denoted 0, 1, ... on the spectrum) is now locally violated whenever two branches merge together (this possibility was noted before in [5]). This behaviour is a result of the strong Coriolis shift (influenced both by rotation and the curvature $B_\theta$), which gives the unstable modes a large non-zero oscillation frequency. Corresponding eigenfunctions of the hybrid modes peak near the inner wall, whereas the slow instabilities are more confined internally.

We do not show a spectrum for $m \neq 0$ but note that similar behaviour appears where discrete modes go from oscillating to unstable passing over the continua, although the interactions of different continuum ranges complicates the picture. Notably, the unstable eigenmodes are not located above the Doppler range, which is always the case in thin disks [4]. This could also be a result of the strong Coriolis shift, which dominates over the Doppler shift in this case. In a simpler case with uniform rotation and constant pitch, all continua collapse to a single point and it was explicitly shown that even $m \neq 0$ discrete modes obey a similar phenomenology as the $m = 0$ modes in Fig. 1 [6]. In terms of eigenfunctions, the Doppler corotation radius or
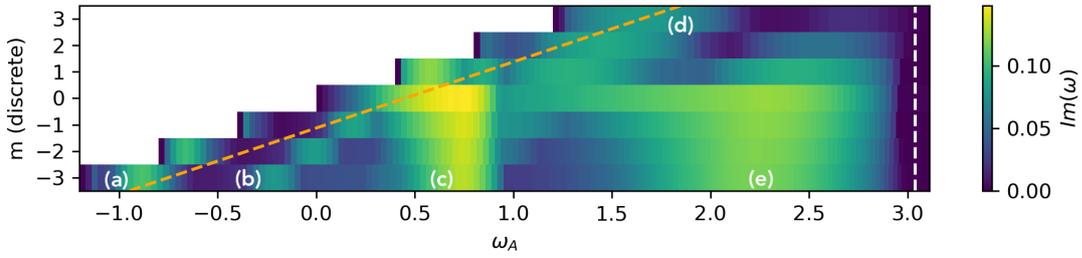
Figure 2: *Growth rates of the most unstable modes over a range of wavenumbers m and k. The dashed orange line denotes the smallest wavenumber k = 45 that fits into the scale height. The dashed white line shows the PP05 [3] instability criterion, which we generalise here for non-zero m values. Distinct branches are labeled (a)–(e). Note how additional branches of hybrid or slow unstable modes appear.* resonances with the continua play a limited role in confining the oscillatory parts.

With this information, we extend the study of [2] regarding maximal growth rates to non-axisymmetric instabilities. Figure 2 features the growth rates of the most unstable mode for multiple values of $m$. The $m = 0$ case reproduces their results that two branches of unstable modes compete: the most unstable mode dominating at lower $k$ is a hybrid slow-Alfvén mode, and slow-like modes dominating at higher $k$. Growth is then cut off when $kB_z$ approaches the upper bound given by criterion (1). For $m \neq 0$, this picture complicates significantly, with several additional branches appearing, but the same instability criterion $\omega_A < k_c^2$ remains valid. Further study is required to determine whether a local dispersion relation as that in [7] can be used to explicitly obtain this criterion for $m \neq 0$. The branches are labeled in Fig. 2 according to their nature: branches (a), (b) and (c) result from the interaction (overlap) of one or both Alfvén continua with the slow continua, and the unstable modes are located above either. This allows to interpret them as the generalisations of the hybrid instabilities of [2]. For branches (d) and (e), only the slow continua overlap, and these would then generalise the slow mode instability of [2]. Branch (c) and (d) overlap for $m < 0$. In terms of growth rates, the $m < 0$ instabilities are comparable to the $m = 0$ instabilities.

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