

MHD wave modeling with Q-variables

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The Elsässer variables have been widely used in magnetohydrodynamics (MHD) to describe Alfvén waves. However, Elsässer variables are less suited for other wave modes, such as fast and slow magnetosonic waves, which require mixed components. To address this limitation, Q-variables were introduced as a generalisation of the Elsässer formalism, which allows for tracking waves with phase speeds different from the Alfvén speed. This extension provides a new framework for studying MHD wave dynamics.

In this work, we extend the Q-variable framework by analytically deriving the dispersion relation for magnetosonic waves. We demonstrate that the Q-formalism is complete and consistent for compressive waves, and we obtain an explicit expression for the parameter α required to separate wave modes in a reference frame where $\omega^- = 0$. These results provide the theoretical foundation for incorporating compressive wave drivers in solar wind models such as UAWSOM (Uniturbulence and Alfvén Wave Solar Models).

1 Introduction

MHD waves are believed to play a crucial role in heating and accelerating the solar wind. The Elsässer variables [1] are used in turbulence studies and solar wind modeling, particularly in wave-driven models such as AWSOM (Alfvén Wave Solar Models) [2]. However, the utility of Elsässer variables is limited to pure Alfvén waves and breaks down when applied to compressive MHD wave modes, such as fast and slow magnetosonic waves, which typically involve mixed contributions from both Elsässer components [3, 4]. To overcome this limitation, Q-variables were recently introduced by Van Doorselaere et al. [5] as a generalization of Elsässer variables:

$$Q^\pm = V \pm \alpha B \quad (1)$$

where V is the speed of the plasma, B is the magnetic field, and α is a new parameter, which describes the wave phase speed for a general wave. In the $\alpha = 1/\sqrt{\mu\rho}$ limit, we recover the Elsässer variables $Z^\pm = V \pm V_A$, where V_A is an Alfvén velocity.

In the initial study [5], the Q-variables were shown to correctly describe Alfvén and kink

waves and were subsequently applied as a foundation for the UAWSOM framework [6], which extends AWSOM by including the non-linear damping of kink waves [4, 7].

However, the derivation of dispersion relations for magnetosonic modes was not explicitly done in the original Q-variable framework. In this work, we address that gap. We derive the dispersion relation directly from the linearised Q-equations and identify the necessary form of α that allows the separation of slow and fast magnetosonic waves. This result confirms the internal consistency of the Q-variable system and enables its direct application in wave-driven models like UAWSOM, which allows for accurate tracking of compressive wave energy in the solar atmosphere.

2 Results

We start from the reformulated MHD equations in terms of Q-variables, as introduced in [5]. The linear wave solutions are expressed using the standard plane wave notation $\exp[i(\mathbf{k} \cdot \mathbf{x} - \omega t)]$, where we choose the x -axis to be in the \mathbf{k} - \mathbf{B}_0 plane. This choice implies $k_y = 0$. For plane waves, the co-moving derivative becomes $\frac{d^\pm}{dt} = -i(\omega - \mathbf{k} \cdot \mathbf{Q}_0^\pm) \equiv -i\omega^\pm$. The linearised Q-variable equations then reduce to the following system:

$$-\omega^- \delta R = -k_x \delta Q_x^- - k_z \delta Q_z^- - \alpha B_0 k_z \delta R \quad (2)$$

$$-\omega^+ \delta R = -k_x \delta Q_x^+ - k_z \delta Q_z^+ + \alpha B_0 k_z \delta R \quad (3)$$

$$-\omega^- \delta Q_x^+ = -v_{s0}^2 k_x \delta R - \left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right) \frac{\alpha B_0 k_x}{2} (\delta Q_z^+ - \delta Q_z^-) - \frac{\Delta\alpha^2}{\alpha} \frac{B_0 k_z}{2} (\delta Q_x^+ - \delta Q_x^-) \quad (4)$$

$$-\omega^+ \delta Q_x^- = -v_{s0}^2 k_x \delta R - \left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right) \frac{\alpha B_0 k_x}{2} (\delta Q_z^+ - \delta Q_z^-) - \frac{\Delta\alpha^2}{\alpha} \frac{B_0 k_z}{2} (\delta Q_x^+ - \delta Q_x^-) \quad (5)$$

$$-\omega^- \delta Q_z^+ = -v_{s0}^2 k_z \delta R - \alpha B_0 k_z \delta Q_z^+ - \frac{1}{2} \alpha B_0 k_x (\delta Q_x^+ - \delta Q_x^-) \quad (6)$$

$$-\omega^+ \delta Q_z^- = -v_{s0}^2 k_z \delta R + \alpha B_0 k_z \delta Q_z^- + \frac{1}{2} \alpha B_0 k_x (\delta Q_x^+ - \delta Q_x^-) \quad (7)$$

$$k_x (\delta Q_x^+ - \delta Q_x^-) + k_z (\delta Q_z^+ - \delta Q_z^-) = 0 \quad (8)$$

where $\Delta\alpha^2 = \alpha^2 - 1/\mu\rho$. We assume that $|\mathbf{k}| = 1$, so the components become $k_z = \cos \theta$ and $k_x = \sin \theta$, where θ is the angle between the wavevector \mathbf{k} and the background magnetic field $\mathbf{B}_0 = B_0 \mathbf{e}_z$. Additionally, we consider a stationary background with no flow $\mathbf{V}_0 = 0$. Since Eqs. (2-3) are linearly dependent on Eqs. (6-7), we have a system of five equations with five unknowns. By adding and subtracting equation pairs and substituting variables accordingly, the system can be simplified to a single variable. Substituting these new expressions into a

solenoidal constraint (Eqs. 8) yields a biquadratic equation in ω :

$$\omega^4 - \omega^2(v_{s0}^2 + V_{A0}^2) + v_{s0}^2 V_{A0}^2 \cos^2(\theta) = 0, \quad (9)$$

which is the standard dispersion relation for magnetosonic waves, with its solutions

$$\omega = \pm_{ud} \sqrt{\frac{1}{2}(v_{s0}^2 + V_{A0}^2)} \sqrt{1 \pm_{sf} \sqrt{1 - \frac{4v_{s0}^2 V_{A0}^2 \cos^2(\theta)}{(v_{s0}^2 + V_{A0}^2)^2}}}. \quad (10)$$

This confirms that the Q-variable formalism is able to recover compressive wave behavior, the validity of the Q-variable framework [5] and its consistency with classical MHD wave theory. Notably, all terms involving α cancel in the process, as they, together with $\Delta\alpha^2$, reduce to expressions involving the Alfvén speed. While this may seem unexpected, it is a necessary outcome because our goal is to confirm that the system recovers standard physics, rather than deriving α from it. To obtain an expression for α , we treat it as an unknown variable by adding one more equation to the system. We assume the wave is co-moving with the negative Q-frame, so that $\omega^- = 0$. This changes the system structure and introduces α as a sixth unknown, where the modified system is:

$$0 = -k_z \delta Q_z^- - \alpha B_0 k_z \delta R \quad (11)$$

$$-\omega^+ \delta R = -k_x \delta Q_x^+ - k_z \delta Q_z^+ + \alpha B_0 k_z \delta R \quad (12)$$

$$0 = -v_{s0}^2 k_x \delta R - \frac{1}{2} \left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right) \alpha B_0 k_x (\delta Q_z^+ - \delta Q_z^-) - \frac{1}{2} \frac{\Delta\alpha^2}{\alpha^2} \alpha B_0 k_z \delta Q_x^+ \quad (13)$$

$$0 = -v_{s0}^2 k_x \delta R - \frac{1}{2} \left(1 - \frac{\Delta\alpha^2}{\alpha^2}\right) \alpha B_0 k_x (\delta Q_z^+ - \delta Q_z^-) - \frac{1}{2} \frac{\Delta\alpha^2}{\alpha^2} \alpha B_0 k_z \delta Q_x^+ \quad (14)$$

$$0 = -v_{s0}^2 k_z \delta R - \alpha B_0 k_z \delta Q_z^+ - \frac{1}{2} \alpha B_0 k_x \delta Q_x^+ \quad (15)$$

$$-\omega^+ \delta Q_z^- = -v_{s0}^2 k_z \delta R + \alpha B_0 k_z \delta Q_z^- + \frac{1}{2} \alpha B_0 k_x \delta Q_x^+ \quad (16)$$

$$k_x \delta Q_x^+ + k_z (\delta Q_z^+ - \delta Q_z^-) = 0. \quad (17)$$

Following the same procedure on the simplified system (with $\omega^- = 0$), we arrive at the biquadratic equation in α , where its solution is:

$$\alpha^2 = \frac{(V_{A0}^2 + v_{s0}^2) \pm \sqrt{(V_{A0}^2 + v_{s0}^2)^2 - 4V_{A0}^2 v_{s0}^2 \cos^2 \theta}}{2k_z^2 B_0^2} \equiv \frac{\omega^2}{(\mathbf{k} \cdot \mathbf{B}_0)^2}. \quad (18)$$

This matches the form in [5], where:

$$\omega^- = \omega - \mathbf{k} \cdot \mathbf{Q}_0^- = 0 \quad \Rightarrow \quad \alpha = \frac{\omega - \mathbf{k} \cdot \mathbf{V}_0}{\mathbf{k} \cdot \mathbf{B}_0}. \quad (19)$$

In the absence of background flow, we recover the same expression for α , which could also be obtained by setting $\omega^+ = 0$. This confirms that α naturally corresponds to the phase speed of the wave in the chosen Q-frame, which makes it a powerful tool for wave separation and modeling within the Q-variable formalism.

3 Conclusions

In this work, we have extended the Q-variable formalism introduced by Van Doorselaere et al. [5] by analytically deriving the dispersion relation for magnetosonic waves. Starting from the linearised Q-equations and applying a plane wave solution, we demonstrated that the system recovers the standard biquadratic dispersion relation for magnetosonic waves. Importantly, we showed that when α is treated as a fixed parameter, it cancels from the final expression, confirming the consistency of the Q-formalism with classical MHD theory. To extract an expression for α , we imposed the co-moving condition $\omega^- = 0$, and obtained an explicit solution for α .

This result confirms that the Q-variables not only recover Alfvénic behavior but also allow for clean separation of magnetosonic modes when α is chosen appropriately. The derived expression for α aligns with previous results and can be directly applied in wave-based solar wind models such as UAWSOM, which enable more accurate inclusion of compressive wave drivers.

These findings broaden the applicability of the Q-variable formalism and support its use as a unified framework for analyzing and modeling MHD wave dynamics in astrophysical plasmas.

References

- [1] Elsässer, W. M., 1950, “The hydromagnetic equations,” *Phys. Rev.*, 79(1), 183.
- [2] van der Holst, B. *et al.*, 2014, “The Alfvén Wave Solar Model (AWSOM),” *ApJ*, 782, 81.
- [3] Goossens, M. *et al.*, 2012, “The transverse MHD wave spectrum of magnetic structures,” *Space Sci. Rev.*, 158, 289.
- [4] Magyar, N. *et al.*, 2019, “On the uniturbulent damping of kink waves,” *ApJ*, 884, 144.
- [5] Van Doorselaere, T. *et al.*, 2024, “The magnetohydrodynamic equations in terms of waveframe variables,” *J. Plasma Phys.*, 90(1), 905900113.
- [6] Van Doorselaere, T. *et al.*, 2025, “Uniturbulence and Alfvén wave solar model,” *A&A*, 696, A166.
- [7] Ismayilli, A. *et al.*, 2022, “Analytical modeling of surface Alfvén wave damping,” *ApJ*, 936, 36.